Solution Math 301-101 Sec: 02 & 03 Quiz 6 (A)

Q:1 Solve the heat equation

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0 \\ u(0,t) &= 0, \;\; u(\pi,t) = 0, \quad t > 0 \\ u(x,0) &= 100, \quad 0 < x < \pi. \end{aligned}$$

Sol: Let $u(x,t) = X(x)T(t)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \implies X''T = XT' \implies \frac{X''}{X} = \frac{T'}{T} = -\lambda \implies X'' + \lambda X = 0 \; and \; T' + \lambda T = 0$$

$$u(0,t) = 0, \implies X(0) = 0 \; and \; u(\pi,t) = 0 \implies X(\pi) = 0$$

1. For $\lambda = 0, X'' + \lambda X = 0$ has solution $X(x) = c_1 + c_2 x$.

Using $X(0) = 0$ and $X(\pi) = 0$ we get the trivial solution $X(x) = 0$.

2. For $\lambda = -\alpha^2, X'' + \lambda X = 0$ has solution $X(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$.

Using $X(0) = 0$ we get $c_1 = 0$ and using $X(\pi) = 0$ we get $c_2 \sinh(\alpha\pi) = 0$. Since $\sinh(\alpha\pi) = 0$ only if $\alpha = 0$. But $\alpha \neq 0$, therefore $c_2 = 0$ and we get a trivial solution $X(x) = 0$.

3. For $\lambda = \alpha^2, X'' + \lambda X = 0$ has solution $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.

Using $X(0) = 0$ we get $c_1 = 0$ and using $X(\pi) = 0$ we get $c_2 \sin(\alpha\pi) = 0$. For nontrivial solution $c_2 \neq 0$ and $\sin(\alpha\pi) = 0 \implies \alpha\pi = n\pi$ or $\alpha = n, \; n = 1, 2, 3, \ldots$. The solutio is $X(x) = c_2 \sin(nx), \; n = 1, 2, 3, \ldots$.

For $\lambda = \alpha^2 = n^2, T' + n^2 T = 0$ has solution $T(t) = c_3 e^{-n^2 t}$.

The general solution is $u(x,t) = \sum_{i=1}^{\infty} A_n \sin(nx) e^{-n^2 t}$.

$u(x,0) = 100 \implies 100 = \sum_{i=1}^{\infty} A_n \sin(nx)$ which is a half range Fourier sine series.

$A_n = \frac{2}{\pi} \int_0^{\pi} 100 \sin(nx) dx = \frac{-200}{n\pi}((-1)^n - 1) = \frac{200}{n\pi}(1 - (-1)^n)$

So $u(x,t) = \sum_{i=1}^{\infty} \frac{200}{n\pi}(1 - (-1)^n) \sin(nx) e^{-n^2 t}$

Q:2 Solve the wave equation

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0 \\ u(0,t) &= 0, \ \ u(L,t) = 0, \quad t > 0 \\ u(x,0) &= 0, \quad u_t(x,0) = g(x), \quad 0 < x < \text{L}.\end{aligned}$$

Sol: Let $u(x,t) = X(x)T(t)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \implies X''T = XT'' \implies \frac{X''}{X} = \frac{T''}{T} = -\lambda \implies X'' + \lambda X = 0 \ and \ T'' + \lambda T = 0$$

$$u(0,t) = 0, \implies X(0) = 0 \ and \ u(L,t) = 0 \implies X(L) = 0$$

1. For $\lambda = 0, X'' + \lambda X = 0$ has solution $X(x) = c_1 + c_2 x$.

Using $X(0) = 0$ and $X(L) = 0$ we get the trivial solution $X(x) = 0$.

2. For $\lambda = -\alpha^2, X'' + \lambda X = 0$ has solution $X(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$.

Using $X(0) = 0$ we get $c_1 = 0$ and using $X(L) = 0$ we get $c_2 \sinh(\alpha L) = 0$. Since $\sinh(\alpha L) = 0$ only if $\alpha = 0$. But $\alpha \neq 0$, therefore $c_2 = 0$ and we get a trivial solution $X(x) = 0$.

3. For $\lambda = \alpha^2, X'' + \lambda X = 0$ has solution $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.

Using $X(0) = 0$ we get $c_1 = 0$ and using $X(L) = 0$ we get $c_2 \sin(\alpha L) = 0$. For nontrivial solution $c_2 \neq 0$ and $\sin(\alpha L) = 0 \implies \alpha L = n\pi$ or $\alpha = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$. The solutio is $X(x) = c_2 \sin(\frac{n\pi}{L} x)$, $n = 1, 2, 3, \ldots$.

For $\lambda = \alpha^2, T'' + \alpha^2 T = 0$ has solution $T(t) = c_3 cos(\frac{n\pi}{L} t) + c_4 \sin(\frac{n\pi}{L} t)$.

$u(x,0) = 0, \implies T(0) = 0 \implies c_3 = 0$.

The general solution is $u(x,t) = \sum_{i=1}^{\infty} A_n \sin(\frac{n\pi}{L} x) \sin(\frac{n\pi}{L} t)$.

Now $u_t(x,0) = g(x) \implies g(x) = \sum_{i=1}^{\infty} A_n \frac{n\pi}{L} \sin(\frac{n\pi}{L} x)$ which is a half range Fourier sine series

series, where $A_n \frac{n\pi}{L} = \frac{2}{L} \int_0^L g(x) \sin(\frac{n\pi}{L} x) dx$ or $A_n = \frac{2}{n\pi} \int_0^L g(x) \sin(\frac{n\pi}{L} x) dx$.