Solution Math 301-101 Sec: 02 & 03 Quiz 4 (B)

Q:1 Show that the set of functions $\left\{ \sin(\frac{n\pi}{p})x \right\}$ is an orthogonal set on $[0, p]$ for $n = 1, 2, 3, \cdots$. Also find norm of each function. (Justify your answer with reason).

Sol: Let $\phi_n(x) = \sin(\frac{n\pi}{p}x)$. Then for $n \neq m$

$$(\phi_n, \phi_m) = \int_0^p \sin(\tfrac{n\pi}{p}x)\sin(\tfrac{m\pi}{p}x)dx$$
$$= \tfrac{1}{2}\int_0^p \left[\cos(\tfrac{(n-m)\pi}{p}x) - \cos(\tfrac{(n+m)\pi}{p}x)\right]dx$$
$$= \tfrac{1}{2}\left[\frac{p}{(n-m)\pi}\sin(\tfrac{(n+m)\pi}{p}x) - \frac{p}{(n+m)\pi}\sin(\tfrac{(n-m)\pi}{p}x)\right]_0^p = 0.$$

$$\|\phi_n\|^2 = \int_0^p \sin^2(\tfrac{n\pi}{p}x)dx = \tfrac{1}{2}\int_0^p (1-\cos(\tfrac{2n\pi}{p}x))dx = \tfrac{1}{2}\left[x - \tfrac{p}{2n\pi}\sin(\tfrac{2n\pi}{p}x)\right]_0^p = \frac{p}{2}$$

So $\|\phi_n\| = \sqrt{\frac{p}{2}}$

Q:2 Find half range cosine expansion of

$$f(x) = \sin(x), \quad 0 < x < \pi.$$

Sol: $a_0 = \frac{2}{\pi}\int_0^\pi \sin(x)dx = \frac{-2}{\pi}\left[cos(x)\right]_0^\pi = \frac{4}{\pi}$

$$a_n = \tfrac{2}{\pi}\int_0^\pi \sin(x)\cos(\tfrac{n\pi}{\pi}x)dx = \tfrac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)dx$$
$$= \tfrac{1}{\pi}\int_0^\pi \left[\sin((1+n)x) + \sin((1-n)x)\right]dx$$
$$= \tfrac{1}{\pi}\left[\tfrac{-1}{1+n}\cos((1+n)x) + \tfrac{-1}{1-n}\cos((1-n)x)\right]_0^\pi dx$$
$$= \tfrac{1}{\pi}\left[\frac{(-1)^{n+2}}{1+n} - \frac{(-1)^{n-1}}{1-n} + \frac{1}{1+n} - \frac{1}{1-n}\right] = \frac{2(1+(-1)^n)}{\pi(1-n^2)}$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^\infty \frac{2(1+(-1)^n)}{\pi(1-n^2)}\sin(nx)$$

Q:3 Find Fourier series expansion of

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ 1-x & \text{if } 0 \le x < 1 \end{cases}.$$

Sol: $a_0 = \frac{1}{1}\int_{-1}^0 dx + \frac{1}{1}\int_0^5 (1-x)dx = \frac{3}{2}$

$$a_n = \int_{-1}^0 \cos(n\pi)xdx + \int_0^1 (1-x)\cos(n\pi)xdx = \frac{(1-(-1)^n)}{n^2\pi^2}$$

$$b_n = \int_{-1}^0 \sin(n\pi)xdx + \int_0^1 (1-x)\sin(n\pi)xdx = \frac{(-1)^n}{n\pi}$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^\infty \left[\frac{(1-(-1)^n)}{n^2\pi^2}\cos(n\pi)x + \frac{(-1)^n}{n\pi}\sin(n\pi)x\right]$$