Solution Math 301-101 Sec: 02 & 03 Quiz 4 (A)

Q:1 Show that the set of functions $\left\{ \cos(\frac{n\pi}{p})x \right\}$ is an orthogonal set on $[0, p]$ for $n = 1, 2, 3, \cdots$. Also find norm of each function. (Justify your answer with reason).

Sol: Let $\phi_n(x) = \cos(\frac{n\pi}{p}x)$. Then for $n \neq m$

$$(\phi_n, \phi_m) = \int_0^p \cos(\tfrac{n\pi}{p}x)\cos(\tfrac{m\pi}{p}x)dx$$

$$= \tfrac{1}{2}\int_0^p \left[\cos(\tfrac{(n+m)\pi}{p}x) + \cos(\tfrac{(n-m)\pi}{p}x)\right] dx$$

$$= \tfrac{1}{2}\left[\frac{p}{(n+m)\pi}\sin(\tfrac{(n+m)\pi}{p}x) + \frac{p}{(n-m)\pi}\sin(\tfrac{(n-m)\pi}{p}x)\right]_0^p = 0.$$

$$\|\phi_n\|^2 = \int_0^p \cos^2(\tfrac{n\pi}{p}x)dx = \tfrac{1}{2}\int_0^p (1 + \cos(\tfrac{2n\pi}{p}x))dx = \tfrac{1}{2}\left[x + \tfrac{p}{2n\pi}\sin(\tfrac{2n\pi}{p}x)\right]_0^p = \frac{p}{2}$$

So $\|\phi_n\| = \sqrt{\dfrac{p}{2}}$

Q:2 Find half range cosine expansion of

$$f(x) = \cos(x), \quad 0 < x < \frac{\pi}{2}.$$

Sol: $a_0 = \frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos(x)dx = \frac{4}{\pi}\left[sin(x)\right]_0^{\frac{\pi}{2}} = \frac{4}{\pi}$

$$a_n = \tfrac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos(x)\cos(\tfrac{n\pi}{\frac{\pi}{2}}x)dx = \tfrac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x)\cos(2nx)dx$$

$$= \tfrac{2}{\pi}\int_0^{\frac{\pi}{2}} \left[\cos((2n+1)x) + \cos((2n-1)x)\right] dx$$

$$= \tfrac{2}{\pi}\left[\tfrac{1}{2n+1}\sin((2n+1)x) + \tfrac{1}{2n-1}\sin((2n-1)x)\right]_0^{\frac{\pi}{2}} dx$$

$$= \tfrac{2}{\pi}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1}\right] = \frac{4(-1)^n}{\pi(1-4n^2)}$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)}\cos(2nx)$$

Q:3 Find Fourier series expansion of

$$f(x) = \begin{cases} 1 & \text{if } -5 < x < 0 \\ 1 + x & \text{if } 0 \leq x < 5 \end{cases}.$$

Sol: $a_0 = \frac{1}{5}\int_{-5}^0 dx + \frac{1}{5}\int_0^5 (1+x)dx = \frac{9}{2}$

$$a_n = \tfrac{1}{5}\int_{-5}^0 \cos(\tfrac{n\pi}{5})xdx + \tfrac{1}{5}\int_0^5 (1+x)\cos(\tfrac{n\pi}{5})xdx = \frac{5((-1)^n - 1)}{n^2\pi^2}$$

$$b_n = \tfrac{1}{5}\int_{-5}^0 \sin(\tfrac{n\pi}{5})xdx + \tfrac{1}{5}\int_0^5 (1+x)\sin(\tfrac{n\pi}{5})xdx = \frac{5(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{9}{4} + \sum_{n=1}^{\infty}\left[\frac{5((-1)^n - 1)}{n^2\pi^2}\cos(\tfrac{n\pi}{5})x + \frac{5(-1)^{n+1}}{n\pi}\sin(\tfrac{n\pi}{5})x\right]$$