Solution Math 301 Quiz 2 (A)

Q.1: Find the work done by the force $\mathbf{F}(x,y) = (2x + e^{-y})\ \mathbf{i} + (4y - xe^{-y})\ \mathbf{j}$ along the curve $y = x^4$ for $0 \le x \le 1$.

Sol: Let $P(x,y) = 2x + e^{-y}$ and $Q(x,y) = 4y - xe^{-y}$.

$\dfrac{\partial Q}{\partial x} = -e^{-y}$ and $\dfrac{\partial P}{\partial y} = -e^{-y}$, therefore the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.

Using the path $y = x$, we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (2x + e^{-x})\, dx + (4x - xe^{-x})\, dx = \int_0^1 (6x + e^{-x} - xe^{-x})\, dx = e^{-1} + 3.$$

Q.2: Use Green's theorenm to evaluate the integral $\oint_C xydx + x^2 dy$, where C is the boundary of the region determined by the graphs of $x = 0,\ x^2 + y^2 = 1,\ x \ge 0$.

Sol: Let $P(x,y) = xy$ and $Q(x,y) = x^2$. Then using Green's theorem

$$\oint_C xydx + x^2dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R x\ dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r\cos\theta\ rdrd\theta = \frac{1}{3}(2) = \frac{2}{3}.$$

Q.3: Find surface area of that portion of the plane $2x + 3y + 4z = 12$ that is bounded by the coordinate planes in the first octant.

Sol: $z = 3 - \frac{1}{2}x - \frac{3}{4}y$ and $\dfrac{\partial z}{\partial x} = -\dfrac{1}{2},\ \dfrac{\partial z}{\partial y} = -\dfrac{3}{4}$ gives $ds = \sqrt{1 + \dfrac{1}{4} + \dfrac{9}{16}} = \dfrac{\sqrt{29}}{4}$.

$$A(S) = \iint_S dS = \iint_R \frac{\sqrt{29}}{4} dA = \frac{\sqrt{29}}{4}(\text{Area of the Projection onto } xy - plane)$$
$$= \frac{\sqrt{29}}{4}\left(\frac{6.4}{2}\right) = 3\sqrt{29}.$$