Solution Math 301-101 Sec: 02 & 03 Quiz 1 (B)

Q.1: Find tangent vector and equation of tangent line to the curve $\vec{\mathbf{r}}(t) = 2\sin t\ \hat{\imath} + 6\cos t\ \hat{\jmath}$

Sol: Tangent vector $\vec{\mathbf{r}}'(t) = 2\cos t\ \hat{\imath} - 6\sin t\ \hat{\jmath}$

Tangent vector at $t = \frac{\pi}{3}$ is $\vec{\mathbf{r}}'\left(\frac{\pi}{3}\right) = 2\cos\left(\frac{\pi}{3}\right)\ \hat{\imath} - 6\sin\left(\frac{\pi}{3}\right) t\ \hat{\jmath} = \hat{\imath} - 3\sqrt{3}\ \hat{\jmath}$

$\vec{\mathbf{r}}\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{\pi}{3}\right)\ \hat{\imath} + 6\cos\left(\frac{\pi}{3}\right)\ \hat{\jmath} = \sqrt{3}\ \hat{\imath} + 3\ \hat{\jmath}$

Equation of tengent line $\frac{x-\sqrt{3}}{1} = \frac{y-3}{-3\sqrt{3}} = t$ (symmetric form)

$x = t + \sqrt{3}$ and $y = -3\sqrt{3}t + 3$ (parametric form)

Q.2: Find the directional derivative of $f(x,y) = 5x - 3xy + 2y$ at $(2,3)$ in the direction of a vector with angle $\theta = \frac{\pi}{6}$.

Sol: $\nabla f(x,y) = (5-3y)\,\hat{\imath} + (-3x+2)\ \hat{\jmath}$ and $\nabla f(2,3) = -4\ \hat{\imath} - 4\ \hat{\jmath}$

Direction vector is $\vec{\mathbf{u}} = \cos\left(\frac{\pi}{6}\right)\ \hat{\imath} + \sin\left(\frac{\pi}{6}\right)\ \hat{\jmath} = \frac{\sqrt{3}}{2}\ \hat{\imath} + \frac{1}{2}\ \hat{\jmath}$

$D_{\vec{\mathbf{u}}} f(3,-1) = \nabla f(3,-1) \cdot \vec{\mathbf{u}} = -4\frac{\sqrt{3}}{2} - 4\frac{1}{2} = -2\sqrt{3} - 2.$

Q.3: Find the curl and divergence of $\vec{\mathbf{F}}(x,y,z) = xe^{-z}\ \hat{\imath} + 4yz^2\ \hat{\jmath} + 3ze^{-z}\ \hat{\mathbf{k}}$.

Sol: $\nabla \cdot \vec{\mathbf{F}} = \frac{\partial(xe^{-z})}{\partial x} + \frac{\partial(4yz^2)}{\partial y} + \frac{\partial(3ze^{-z})}{\partial z} = e^{-z} + 4z^2 + 3e^{-z} - 3ze^{-z} = 4e^{-z} + 4z^2 - 3ze^{-z}$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{-z} & 4yz^2 & 3ze^{-z} \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial(3ze^{-z})}{\partial y} - \frac{\partial(4yz^2)}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial(xe^{-z})}{\partial z} - \frac{\partial(3ze^{-z})}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial(4yz^2)}{\partial x} - \frac{\partial(xe^{-z})}{\partial y}\right)$$

$$= \hat{\imath}(-8yz) + \hat{\jmath}(-xe^{-z}) + \hat{\mathbf{k}}(0) = -8yz\ \hat{\imath} - xe^{-z}\ \hat{\jmath}$$