Solution Math 301-101 Sec: 02 & 03 Quiz 1 (A)

Q.1: Find tangent vector and equation of tangent line to the curve $\vec{\mathbf{r}}(t) = 3\cos t\ \hat{\imath} + 4\sin t\ \hat{\jmath}$

Sol: Tangent vector $\vec{\mathbf{r}}'(t) = -3\sin t\ \hat{\imath} + 4\cos t\ \hat{\jmath}$

Tangent vector at $t = \frac{\pi}{3}$ is $\vec{\mathbf{r}}'\left(\frac{\pi}{3}\right) = -3\sin\left(\frac{\pi}{3}\right)\ \hat{\imath} + 4\cos\left(\frac{\pi}{3}\right) t\ \hat{\jmath} = -3\frac{\sqrt{3}}{2}\ \hat{\imath} + \mathbf{2}\ \hat{\jmath}$

$\vec{\mathbf{r}}\left(\frac{\pi}{3}\right) = 3\cos\left(\frac{\pi}{3}\right)\ \hat{\imath} + 4\sin\left(\frac{\pi}{3}\right)\ \hat{\jmath} = \frac{3}{2}\ \hat{\imath} + 2\sqrt{3}\ \hat{\jmath}$

Equation of tangent line $\dfrac{x - \frac{3}{2}}{-3\frac{\sqrt{3}}{2}} = \dfrac{y - 2\sqrt{3}}{2} = t$ (symmetric form)

$x = -3\frac{\sqrt{3}}{2}t + \frac{3}{2}$ and $y = 2t + 2\sqrt{3}$ (parametric form)

Q.2: Find the directional derivative of $f(x, y) = 4x + xy^2 - 5y$ at $(3, -1)$ in the direction of a vector with angle $\theta = \frac{\pi}{4}$.

Sol: $\nabla f(x, y) = (4 + y^2)\,\hat{\imath} + (2xy - y)\ \hat{\jmath}$ and $\nabla f(3, -1) = 5\ \hat{\imath} - 11\ \hat{\jmath}$

Direction vector is $\vec{\mathbf{u}} = \cos\left(\frac{\pi}{4}\right)\ \hat{\imath} + \sin\left(\frac{\pi}{4}\right)\ \hat{\jmath} = \frac{\sqrt{2}}{2}\ \hat{\imath} + \frac{\sqrt{2}}{2}\ \hat{\jmath}$

$D_{\vec{\mathbf{u}}} f(3, -1) = \nabla f(3, -1) \cdot \vec{\mathbf{u}} = 5\frac{\sqrt{2}}{2} - 11\frac{\sqrt{2}}{2} = -6\frac{\sqrt{2}}{2} = -3\sqrt{2}.$

Q.3: Find the curl and divergence of $\vec{\mathbf{F}}(x, y, z) = (x - y)^3\ \hat{\imath} + e^{-yz}\ \hat{\jmath} + xye^{3y}\ \hat{\mathbf{k}}$.

Sol: $\nabla \cdot \vec{\mathbf{F}} = \dfrac{\partial (x-y)^3}{\partial x} + \dfrac{\partial e^{-yz}}{\partial y} + \dfrac{\partial xye^{3y}}{\partial z} = 3(x-y)^2 - ze^{-yz}$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x-y)^3 & e^{-yz} & xye^{3y} \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial (xye^{3y})}{\partial y} - \frac{\partial (e^{-yz})}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial (x-y)^3}{\partial z} - \frac{\partial (xye^{3y})}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial (e^{-yz})}{\partial x} - \frac{\partial (x-y)^3}{\partial y}\right)$$

$$= \hat{\imath}\,(xe^{3y} + 3xye^{3y} + ye^{-yz}) + \hat{\jmath}\,(-ye^{3y}) + \hat{\mathbf{k}}\left(3(x-y)^2\right)$$