King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Final Exam
The First Semester of 2010-2011 (101)

Time Allowed: 150 Minutes

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Q:1 (a) (5 points) Find Laplace transform $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ t^2 & \text{if } t \geq 1 \end{cases}.$$

(b) (5 points) Find inverse Laplace transform $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\}$.

Sol: (a) We can write $f(t)$ as $f(t) = t^2\mathcal{U}(t-1)$ and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\mathcal{U}(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} = e^{-s}\mathcal{L}\{t^2+2t+1\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$$

(b) Partial fractions of $\dfrac{1}{s^2(s-1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} = \dfrac{-1}{s} + \dfrac{-1}{s^2} + \dfrac{1}{s-1}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{-1}{s}+\frac{-1}{s^2}+\frac{1}{s-1}\right)\right\}$$
$$= (-1-(t-2)+e^{t-2})\mathcal{U}(t-2)$$

Q:2 (12 points) Let $\vec{F}(x,y,z) = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}$ and D is the region bounded by the concentric spheres $x^2+y^2+z^2=9$ and $x^2+y^2+z^2=4$. Use divergence theorem to evaluate $\iint_S\left(\vec{F}\cdot\hat{n}\right)dS$.

Sol: $\vec{F}(x,y,z) = \dfrac{x\mathbf{i}}{x^2+y^2+z^2} + \dfrac{y\mathbf{i}}{x^2+y^2+z^2} + \dfrac{z\mathbf{i}}{x^2+y^2+z^2}$

$$div\vec{F} = \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2} = \frac{1}{\rho^2}$$

$$\iint_S\left(\vec{F}\cdot\hat{n}\right)dS = \iiint_D div\vec{F}dV = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \tfrac{1}{\rho^2}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\ \int_0^{\pi}\sin(\phi)d\phi\ \int_2^3 d\rho$$

$$= 2\pi(1+1)(3-2) = 4\pi.$$

Q:3 (16 points) Use separation of variables method to find the nontrivial solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0 \\ u(x,0) &= 10, \quad 0 < x < 1. \end{aligned}$$

Sol: Let $u(x,t) = X(x)T(t)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \implies X''T = XT' \implies \frac{X''}{X} = \frac{T'}{T} = -\lambda \implies X'' + \lambda X = 0 \; and \; T' + \lambda T = 0$$

$$u(0,t) = 0, \implies X(0) = 0 \; and \; u(1,t) = 0 \implies X(1) = 0$$

For $\lambda = \alpha^2$, $X'' + \lambda X = 0$ has solution $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.

Using $X(0) = 0$ we get $c_1 = 0$ and using $X(1) = 0$ we get $c_2 \sin(\alpha) = 0$. For nontrivial

solution $c_2 \neq 0$ and $\sin(\alpha) = 0 \implies \alpha = n\pi, \; n = 1, 2, 3, \ldots$. The solution is

$X(x) = c_2 \sin(n\pi x), \; n = 1, 2, 3, \ldots$.

For $\lambda = \alpha^2 = n^2\pi^2, T' + \alpha^2 T = 0$ has solution $T(t) = c_3 e^{-n^2\pi^2 t}$.

The general solution is $u(x,t) = \sum_{i=1}^{\infty} A_n \sin(n\pi x) e^{-n^2\pi^2 t}$.

$u(x,0) = 10 \implies 10 = \sum_{i=1}^{\infty} A_n \sin(n\pi x)$ which is a half range Fourier sine series.

$A_n = \frac{2}{1}\int_0^1 10 \sin(n\pi x)dx = \frac{-20}{n\pi}((-1)^n - 1) = \frac{20}{n\pi}(1 - (-1)^n)$

So $u(x,t) = \sum_{i=1}^{\infty} \frac{20}{n\pi}(1 - (-1)^n) \sin(n\pi x) e^{-n^2\pi^2 t}$

Q:4 (20 points) Use separation of variables method to find the nontrivial solution of the wave equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 2, \; t > 0$$

subject to the boundary conditions

$$\begin{aligned} u(2,t) &= 0, \; t > 0 \\ u(r,0) &= 1, \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < r < 2 \\ \text{solution is bounded at } r &= 0. \end{aligned}$$

Sol: Let $u(r,t) = R(r)T(t)$, then $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2} \implies R''T + \frac{1}{r}R'T = RT''$

Dividing by RT, we get $\dfrac{R'' + \frac{1}{r}R'}{R} = \dfrac{T''}{T} = -\lambda \implies rR'' + R' + \lambda rR = 0$ and $T'' + \lambda T = 0$

The equation $rR'' + R' + \lambda rR = 0$ is a parametric Bessel equation with parameter $\lambda = \alpha^2$. Its general solution is given as $R(r) = c_1 J_0(\alpha r) + Y_0(\alpha r)$

Since $Y_0(\alpha r) \to -\infty$ as $r \to 0^+$, therefore we choose $c_2 = 0$ to keep the solution bounded.

The boundary condition $u(2,t) = 0 \implies R(2) = 0$ and $R(2) = 0 \implies c_1 J_0(2\alpha) = 0$. For nontrivial solution we let $c_1 \neq 0$ and therefore $J_0(2\alpha) = 0$.

Let $x_n = 2\alpha_n$ be the positive roots of $J_0(2\alpha) = 0$, $n = 1, 2, 3, \ldots$. Then $R_n(r) = c_1 J_0(\alpha_n r)$ are the solution for $n = 1, 2, 3, \ldots$.

For $\lambda_n = \alpha_n^2$, the solutions of $T'' + \lambda T = 0$ are $T_n(t) = c_3 \cos(\alpha_n t) + c_4 \sin(\alpha_n t)$

The general solution of the original problem is

$u(r,t) = \sum_{i=1}^{\infty} \left[A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right] J_0(\alpha_n r)$

and $u_t(r,t) = \sum_{i=1}^{\infty} \left[-A_n\alpha_n \cos(\alpha_n t) + B_n\alpha_n \sin(\alpha_n t)\right] J_0(\alpha_n r)$

$u_t(r,0) = 0 \implies B_n = 0$ and $u(r,0) = 1 \implies 1 = \sum_{i=1}^{\infty} A_n J_0(\alpha_n 2)$, which is a Fourier-Bessel series, where

$$A_n = \frac{2}{4J_1^2(2\alpha_n)} \int_0^2 rJ_0(\alpha_n r)dr = \frac{1}{2J_1^2(2\alpha_n)} \frac{1}{\alpha_n^2} \int_0^{2\alpha_n} \frac{d}{dt}[tJ_1(t)] = \frac{1}{\alpha_n J_1(2\alpha_n)} dt$$

So $u(r,t) = \sum_{i=1}^{\infty} \dfrac{\cos(\alpha_n t)}{\alpha_n J_1(2\alpha_n)} J_0(\alpha_n r)$.

Q:5 (20 points) Find the steady-state temperature $u(r,\theta)$ in a sphere of radius 2 by solving the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,\quad 0<r<2,\ 0<\theta<\pi,$$

subject to the boundary condition

$$u(2,\theta) \;=\; 1+\cos(\theta),\quad 0<\theta<\pi.$$

Sol: Let $u(r,\theta)=R(r)\Theta(\theta)$, then

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0 \implies R''\Theta+\frac{2}{r}R'\Theta+\frac{1}{r^2}\Theta''R+\frac{\cot(\theta)}{r^2}\Theta'=0$$

Dividing by $R\Theta$, we get $\dfrac{r^2R''+2rR'}{R}=\dfrac{-\Theta''-\cot(\theta)\Theta'}{\Theta}=\lambda.$

The equation $\Theta''+\cot(\theta)\Theta'+\lambda\Theta=0 \implies \sin(\theta)\Theta''+\cos(\theta)\Theta'+\lambda\sin(\theta)\Theta=0.$

Letting $x=\cos(\theta)$, $0<\theta<\pi$, we get $(1-x^2)\dfrac{\partial^2\Theta}{\partial x^2}-2x\dfrac{\partial\Theta}{\partial x}+\lambda\Theta=0.$

This is a Legendre equation with $\lambda=n(n+1)$, $n=0,1,2,\ldots.$

Its solutions are given by $\Theta(\theta)=P_n(\cos(\theta))$.

The other equation $r^2R''+2rR'-\lambda R=0$ is a Cauch-Euler equation. For $\lambda=n(n+1)$, the auxiliary equation is $m(m-1)+2m-n(n+1)=0 \implies m=n,-(n+1)$. The solutions

$R(r)=c_1r^n+c_2r^{-(n+1)}=c_1r^n+\dfrac{c_2}{r^{(n+1)}}\to\infty$ as $r\to 0$, therefore we choose $c_2=0$.

The general solution is $u(r,\theta)=\sum_{i=0}^{\infty}A_nr^nP_n(\cos(\theta))$.

$u(2,\theta)=1+\cos(\theta) \implies 1+\cos(\theta)=\sum_{i=0}^{\infty}A_n2^nP_n(\cos(\theta))$, a Fourier-Legendre series, where

$$A_n=\frac{2n+1}{2^{n+1}}\int_0^\pi(1+\cos(\theta))P_n(\cos(\theta))\sin(\theta)d\theta$$

$$=\frac{2n+1}{2^{n+1}}\int_0^\pi P_n(\cos(\theta))\sin(\theta)d\theta+\frac{2n+1}{2^{n+1}}\int_0^\pi\cos(\theta)P_n(\cos(\theta))\sin(\theta)d\theta$$

The first integral is equal to zero for all $n \neq 0$ and the second integral is zero for all $n \neq 1$.

$$A_0 = \frac{1}{2}\int_0^\pi P_0(\cos(\theta))\sin(\theta)d\theta = \frac{1}{2}\int_0^\pi \sin(\theta)d\theta = 1$$

$$A_1 = \frac{3}{4}\int_0^\pi \cos(\theta)P_1(\cos(\theta))\sin(\theta)d\theta = \frac{3}{4}\int_0^\pi \cos^2(\theta)\sin(\theta)d\theta = \frac{1}{2}$$

$$u(r,\theta) = 1P_0(\cos(\theta)) + \frac{1}{2}rP_1(\cos(\theta)) = 1 + \frac{1}{2}r\cos(\theta)$$

Q:6 (15 points) Use Laplace transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \ t > 0$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0 \\ u(x,0) &= 2\sin(\frac{\pi x}{L}), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L. \end{aligned}$$

Sol: Laplace transform of the given equation is

$$\frac{d^2U}{dx^2} = s^2U - su(x,0) - u_t(x,0) = s^2U - 2s\sin(\tfrac{\pi x}{L})$$

$$\Longrightarrow \frac{d^2U}{dx^2} - s^2U = -2s\sin(\tfrac{\pi x}{L})$$

$$\Longrightarrow U_c(x,s) = c_1\cosh(sx) + c_2\sinh(sx).$$

For U_p, let $U_p = A\sin(\frac{\pi x}{L}) + B\cos(\frac{\pi x}{L})$, then $U_p'' = -A\frac{\pi^2}{L^2}\sin(\frac{\pi x}{L}) - B\frac{\pi^2}{L^2}\cos(\frac{\pi x}{L})$

$$\Longrightarrow -A\frac{\pi^2}{L^2}\sin(\tfrac{\pi x}{L}) - B\frac{\pi^2}{L^2}\cos(\tfrac{\pi x}{L}) - As^2\sin(\tfrac{\pi x}{L}) - Bs^2\cos(\tfrac{\pi x}{L}) = -2s\sin(\tfrac{\pi x}{L})$$

$\Longrightarrow A = \dfrac{2s}{s^2 + \frac{\pi^2}{L^2}}$ and $B = 0$. So $U(x,s) = c_1\cosh(sx) + c_2\sinh(sx) + \dfrac{2s}{s^2 + \frac{\pi^2}{L^2}}\sin(\frac{\pi x}{L})$

$u(0,t) = 0 \Longrightarrow U(0,s) = 0$ and $u(L,t) = 0 \Longrightarrow U(L,s) = 0$

$U(0,s) = 0 \Longrightarrow c_1 = 0$ and $U(L,s) = 0 \Longrightarrow c_2\sinh(sL) = 0$. Since $\sinh(sL) = 0$ only for $sL = 0$. Since $sL \neq 0$, therefore $c_2 = 0$.

$$U(x,s) = \frac{2s}{s^2+\frac{\pi^2}{L^2}}\sin(\tfrac{\pi x}{L}) \implies u(x,t) = 2\cos(\tfrac{\pi}{L}t)\sin(\tfrac{\pi}{L}x)$$

Q:7 (15 points) Use appropriate Fourier transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad x>0,\ t>0$$

subject to the conditions

$$\begin{aligned} u\,(0,t) &= 5, t>0 \\ u\,(x,0) &= 0, \quad x>0. \end{aligned}$$

Sol: Fourier transform of given equation is

$$-\alpha^2 U(\alpha,t) + \alpha u(0,t) = \frac{dU}{dt} \implies \frac{dU}{dt} + \alpha^2 U = 5\alpha.$$

$U_c = c_1 e^{-\alpha^2 t}$. Let $U_p = A$, then $U_p' = 0$ and $0 + \alpha^2 A = 5\alpha \implies A = \dfrac{5}{\alpha}$

So $U(\alpha,t) = c_1 e^{-\alpha^2 t} + \dfrac{5}{\alpha}$

$$u(x,0) = 0 \implies U(\alpha,0) = 0 \implies 0 = c_1 + \frac{5}{\alpha} \implies c_1 = -\frac{5}{\alpha}$$

$$U(\alpha,t) = \frac{5}{\alpha}\left[1 - e^{-\alpha^2 t}\right] \implies u(x,t) = \tfrac{2}{\pi}\int_0^\infty \tfrac{5}{\alpha}\left[1 - e^{-\alpha^2 t}\right]\sin(\alpha x)d\alpha$$