King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Final Exam
The Summer Semester of 2009-2010 (093)

Q:1 (12 points) Write out the first four nonzero terms in the Fourier-Legendre expansion of

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}.$$

Sol: The first 5 Legendre polynomials are $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

The Fourier-Legendre expansion of $f(x)$ is $f(x) = \sum_{n=0}^{\infty} A_n P_n(x)$, where $A_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$

$$A_0 = \frac{1}{2}\int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2}\int_{0}^{1} x dx = \frac{1}{4},\ A_1 = \frac{3}{2}\int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2}\int_{0}^{1} x^2 dx = \frac{1}{2}$$

$$A_2 = \frac{5}{2}\int_{-1}^{1} f(x) P_2(x)\, dx = \frac{5}{4}\int_{0}^{1} x(3x^2-1)\, dx = \frac{5}{16},\ A_3 = \frac{7}{2}\int_{-1}^{1} f(x) P_3(x)\, dx = \frac{7}{4}\int_{0}^{1} x(5x^3-3x)\, dx = 0$$

$$A_4 = \frac{9}{2}\int_{-1}^{1} f(x) P_4(x)\, dx = \frac{9}{16}\int_{0}^{1} x(35x^4 - 30x^2 + 3)\, dx = -\frac{3}{32}$$

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x).$$

Q:2 (15 points) Find the nontrivial solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2},\quad 0 < x < 1,\quad t > 0$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0,\quad u(1,t) = 0,\quad t > 0 \\ u(x,0) &= \frac{\pi}{2}\sin(3\pi x),\quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\quad 0 < x < 1. \end{aligned}$$

Sol: Let $u(x,t) = X(x)T(t)$, then $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \Rightarrow X''(x)T(t) = \frac{1}{a^2}X(x)T''(t) \Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

$$\Rightarrow X''(x) + \lambda X(x) = 0 \text{ and } T''(t) + a^2\lambda T(t) = 0$$

Let $\lambda = \alpha^2, \alpha > 0$. Then $X''(x) + \lambda X(x) = 0 \;\Rightarrow X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$u(0,t) = 0 \;\Rightarrow c_1 = 0$ and $u(1,t) = 0 \;\Rightarrow c_2 \sin(\alpha) = 0 \;\Rightarrow \alpha = n\pi$ and $c_2 \neq 0$

$X_n(x) = c_2 \sin(n\pi x)$.

Now $T''(t) + a^2\lambda T(t) = 0 \Rightarrow T(t) = c_3 \cos(an\pi t) + c_4 \sin(an\pi t)$ and $T'(t) = -c_3 an\pi \sin(an\pi t) + c_4 n\pi \cos(an\pi t)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \Rightarrow T'(0) = 0 \Rightarrow c_4 = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cos(an\pi t)$$

$$u(x,0) = \frac{\pi}{2}\sin(3\pi x) \;\Rightarrow \frac{\pi}{2}\sin(3\pi x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$\Rightarrow A_n = \frac{2}{1}\int_0^1 \frac{\pi}{2} \sin(3\pi x)\sin(n\pi x) = 0$ for $n \neq 3$ because $\{\sin(n\pi x)\}_{n=1}^{\infty}$ are orthogonal functions.

For $n = 3$, $A_n = \pi \int_0^1 \sin(3\pi x)\sin(3\pi x) = \frac{1}{2}\pi$

$$u(x,t) = \frac{\pi}{2}\sin(3\pi x)\cos(3a\pi t).$$

Q:3 (15 points) Solve the boundary value proble

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < c, \; t > 0$$

subject to the boundary conditions

$$\begin{aligned} u(c,t) &= 0, \; t > 0 \\ u(r,0) &= 0, \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = 1, \quad 0 < r < c \\ \text{solution is bounded at } r &= 0. \end{aligned}$$

Sol: Let $u(r,t) = R(r)T(t)$, then $\frac{\partial^2 u}{\partial r^2} = R''(r)T(t)$, $\frac{\partial u}{\partial r} = R'(r)T(t)$, and $\frac{\partial^2 u}{\partial t^2} = R(r)T''(t)$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \Rightarrow R''(r)T(t) + \frac{1}{r}R'(r)T(t) = \frac{1}{a^2}R(r)T''(t)$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} = \frac{T''(t)}{a^2 T(t)} = -\lambda \Rightarrow R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = 0 \text{ and } T''(t) + a^2\lambda T(t) = 0.$$

$R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = 0 \;\Rightarrow r^2 R''(r) + rR'(r) + \lambda r^2 = 0$ is a Bessel equation with $\nu = 0$.

For $\lambda = \alpha^2$, the general solution of this equation is $R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$

Since $Y_0(\alpha r) \to -\infty$ as $r \to 0^+$, therefore we choose $c_2 = 0$

$u(c,t) = 0 \Rightarrow c_1 J_0(\alpha c) = 0$. For nontrivial solution $c_2 \neq 0$ and $J_0(\alpha c) = 0$

Let $x_i = \alpha_i c$ are non negative solutions of $J_0(\alpha c) = 0$.

Then $\lambda_i = \alpha_i^2 = \dfrac{x_i^2}{c^2}$ are the eigenvalues. $\lambda = 0$ is not an eigenvalues.

Now $T''(t) + a^2\alpha^2 T(t) = 0 \Rightarrow T(t) = c_3 \cos(a\alpha t) + c_4 \sin(a\alpha t)$

$u(r,0) = 0 \Rightarrow T(0) = 0 \Rightarrow c_3 = 0.$

The general solution is $u(r,t) = \sum_{i=1}^{\infty} A_i \sin(a\alpha_i t) J_0(\alpha_i r)$ and $\dfrac{\partial u}{\partial t} = \sum_{i=1}^{\infty} a\alpha_i A_i \cos(a\alpha_i t) J_0(\alpha_i r)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 1 \Rightarrow 1 = \sum_{i=1}^{\infty} a\alpha_i A_i J_0(\alpha_i r),$$

where
$$A_i = \frac{2}{a\alpha_i c^2 J_1^2(\alpha_i c)} \int_0^c r J_0(\alpha_i r)\, dr = \frac{2}{a\alpha_i c^2 J_1^2(\alpha_i c)\,\alpha_i^2} \int_0^{\alpha_i c} t J_0(t)\, dt \quad \text{using } \alpha_i r = t$$

$$= \frac{2}{a\alpha_i c^2 J_1^2(\alpha_i c)\,\alpha_i^2} \int_0^{\alpha_i c} \frac{d}{dt}\left[t J_1(t)\right] dt \quad \text{using } \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \text{ for } n = 1$$

$$= \frac{2}{a\alpha_i c^2 J_1^2(\alpha_i c)\,\alpha_i^2} \left[t J_1(t)\right]\Big|_0^{\alpha_i c} = \frac{2}{a\alpha_i c^2 J_1^2(\alpha_i c)\,\alpha_i^2} \left[\alpha_i c J_1(\alpha_i c)\right]$$

$$= \frac{2}{ac J_1(\alpha_i c)\,\alpha_i^2}$$

$$u(r,t) = \sum_{i=1}^{\infty} \frac{2}{ac J_1(\alpha_i c)\,\alpha_i^2} \sin(a\alpha_i t) J_0(\alpha_i r)$$

Q:4 (8 points) Solution of the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, \quad 0 < r < c, \quad 0 < \theta < \pi$$

is given by

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

Find value(s) of A_n if $u(c,\theta) = \cos\theta$. (Hint: $P_1(\cos\theta) = \cos\theta$).

Sol: $u(c,\theta) = \cos\theta \Rightarrow \cos\theta = \sum_{n=0}^{\infty} A_n c^n P_n(\cos\theta),$

where $A_n c^n = \frac{2n+1}{2}\int\limits_0^\pi \cos\theta P_n(\cos\theta)\sin\theta dd = \frac{2n+1}{2}\int\limits_0^\pi P_1(\cos\theta)\, P_n(\cos\theta)\sin\theta d\theta = 0$
for $n \neq 1$ since $P_n(x)$ are orthogonal

For $n = 1,\ A_1 c = \frac{3}{2}\int\limits_0^\pi P_1(\cos\theta)\, P_1(\cos\theta)\sin\theta d\theta = \frac{3}{2}\int\limits_0^\pi \cos^2\theta\sin\theta d\theta = 1 \Rightarrow A_1 = \frac{1}{c}$

$u(r,\theta) = \frac{1}{c} r\cos\theta.$

Q:5 (5 points) If for $x = \cos\theta,\ P_0(\cos\theta) = 1,\ P_1(\cos\theta) = \cos\theta$. Show that

$$P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1).$$

Sol: $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\Rightarrow P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{2}\left(3\frac{1+\cos 2\theta}{2} - 1\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{3\cos 2\theta}{2} - 1\right) = \frac{1}{4}(3\cos 2\theta + 1).$

Q:6 (15 points) Use Laplace transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \sin\pi x \sin\omega t = \frac{\partial^2 u}{\partial t^2},\quad 0 < x < 1,\ t > 0$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0 \\ u(x,0) &= 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < 1. \end{aligned}$$

Sol: Taking Laplace transform of $\frac{\partial^2 u}{\partial x^2} + \sin\pi x\sin\omega t = \frac{\partial^2 u}{\partial t^2}$ we get

$\frac{d^2 U(x,s)}{dx^2} + \sin\pi x \frac{\omega}{s^2+\omega^2} = s^2 U(x,s) - su(x,0) - u_t(x,0)$

$\frac{d^2 U}{dx^2} - s^2 U = \sin\pi x \frac{\omega}{s^2+\omega^2}$

The complementary function is $U_c(x,s) = c_1\cosh(sx) + c_2\sinh(sx)$

$u(0,t) = 0 \Rightarrow U(0,s) = 0 \Rightarrow c_1 = 0$

$u(1,t) = 0 \Rightarrow U(1,s) = 0 \Rightarrow c_2\sinh(s) = 0 \Rightarrow c_2 = 0$

Let $U_p = A\cos\pi x + B\sin\pi x$, then $\frac{d^2 U_p}{dx^2} = -\pi^2 A\cos\pi x - \pi^2 B\sin\pi x$

$\frac{d^2 U}{dx^2} - s^2 U = \sin\pi x \frac{\omega}{s^2+\omega^2} \Rightarrow -\pi^2 A\cos\pi x - \pi^2 B\sin\pi x - s^2 A\cos\pi x - s^2 B\sin\pi x =$