**King Fahd University of Petroleum & Minerals**
**Department of Mathematics & Statistics**
**SOLUTION Math 301 Major Exam 2**
**The Third Semester of 2012-2013 (123)**

**<u>Time Allowed</u>: 120 Minutes**

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Name: ________________________________ ID#: ____________________

Instructor: ____________________________ Sec #: __________ Serial #: __________

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- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

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| Question # | Marks | Maximum Marks |
|---|---|---|
| 1 | | 14 |
| 2 | | 14 |
| 3 | | 14 |
| 4 | | 18 |
| 5 | | 12 |
| 6 | | 12 |
| 7 | | 16 |
| Total | | 100 |

**Q:1** (7+7 points) Find the following Laplace transforms:

**(a)** $\mathcal{L}\{te^{-t}\sin^2 3t\}$,

**(b)** $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+2s+3)}\right\}$.

**Sol:** (a) $\mathcal{L}\{te^{-t}\sin^2 3t\} = \mathcal{L}\left\{\dfrac{te^{-t}(1-\cos 6t)}{2}\right\} = \mathcal{L}\left\{\dfrac{te^{-t}}{2} - \dfrac{te^{-t}\cos 6t}{2}\right\}$

$$= \frac{1}{2(s+1)^2} + \frac{1}{2}\frac{d}{ds}\left\{\frac{s+1}{(s+1)^2+36}\right\} = \frac{1}{2(s+1)^2} - \frac{1}{2}\left\{\frac{s^2+2s-35}{(s^2+2s+37)^2}\right\}$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+2s+3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s+1)^2+2}\right\} = \dfrac{1}{\sqrt{2}}\sin\sqrt{2}(t-2)e^{-(t-2)}\mathcal{U}(t-2)$

**Q:2** (14 points) Solve the integro-differential equation:

$$f(t) = 2t - \int_0^t (e^{2\tau} - e^{-2\tau})f(t-\tau)d\tau$$

**Sol:** $F(s) = \dfrac{2}{s^2} - \left(\dfrac{1}{s-2} - \dfrac{1}{s+2}\right)F(s)$

$$F(s)\left(1+\frac{4}{s^2-4}\right) = \frac{2}{s^2} \quad \Rightarrow \quad F(s)\left(\frac{s^2}{s^2-4}\right) = \frac{2}{s^2} \quad \Rightarrow F(s) = \frac{2(s^2-4)}{s^4}$$

$$\Rightarrow F(s) = \frac{2}{s^2} - \frac{8}{s^4} \quad \Rightarrow \quad f(t) = 2t - \frac{4}{3}t^3$$

**Q:3** (14 points) Solve the initial value problem using Laplace transform $y'' - y' - 2y = \delta(t-\pi)$ with $y(0) = 1$, $y'(0) = 1$.

**Sol:** $s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) - 2Y(s) = e^{-\pi s}$

$$(s^2-s-2)Y(s) = e^{-\pi s} + s \quad \Rightarrow \quad Y(s) = \frac{e^{-\pi s}}{(s-2)(s+1)} + \frac{s}{(s-2)(s+1)}$$

$$Y(s) = e^{-\pi s}\left(\frac{1}{3(s-2)} - \frac{1}{3(s+1)}\right) + \frac{2}{3(s-2)} + \frac{1}{3(s+1)}$$

$$y(t) = \frac{1}{3}e^{2(t-\pi)}\mathcal{U}(t-\pi) - \frac{1}{3}e^{-(t-\pi)}\mathcal{U}(t-\pi) + \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t}$$

**Q:4** (3+3+12 points) Let $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$.

(a) Sketch the graph of the function $f$.

(b) Sketch the graph of the Fourier series of $f$ on separate coordinate axes.

(c) Find the Fourier series of $f$.

**Sol:** $a_0 = \int_{-1}^{1} f(x)dx = \int_{0}^{1} x\ dx = \dfrac{1}{2}$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\ dx = \int_{0}^{1} x\ \cos n\pi x\ dx = \frac{x\ \sin n\pi x}{n\pi}\ |_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\ dx = \frac{(-1)^n - 1}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\ dx = \int_{0}^{1} x\ \sin n\pi x\ dx = \frac{-x\ \cos n\pi x}{n\pi}\ |_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\ dx = \frac{-(-1)^n}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right)$$

**Q:5** (12 points) Let $f(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & 2 \le x < 3 \end{cases}$.

Find the Fourier sine series of $f$.

**Sol:** $b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{n\pi}{3}x\ dx = \frac{2}{3}\int_0^2 x\ \sin\frac{n\pi}{3}x\ dx = \frac{2}{3}\frac{-3x\ \cos\frac{n\pi}{3}x}{n\pi}\ |_0^2 + \frac{2}{3}\int_0^2 \frac{3\cos\frac{n\pi}{3}x}{n\pi}\ dx$

$$= -\frac{4\cos\frac{2n\pi}{3}}{n\pi} + \frac{2}{3}\frac{9\sin\frac{2n\pi}{3}}{n^2\pi^2}$$

$$f(x) = \sum_{n=1}^{\infty}\left(-\frac{4\cos\frac{2n\pi}{3}}{n\pi} + \frac{6\sin\frac{2n\pi}{3}}{n^2\pi^2}\right)\sin\frac{n\pi}{3}x$$

**Q:6** (6+6 points) Consider the differential equation $y'' - 2y' = \lambda y,\ x \in (0,1)$

with the boundary conditions $y(0) = 0$ and $y(1) = 0$.

(a) Write the differential equation in self-adjoint form.

(b) Is this a regular Sturm–Liouville problem? If yes, write the weight function and associated inner product (orthogonality condition).

**Sol:** (a) The integrating factor is $IF = e^{\int -2dx} = e^{-2x}$

$$e^{-2x}y'' - 2e^{-2x}y' - \lambda e^{-2x}y = 0 \ \Rightarrow\ \frac{d}{dx}\left(e^{-2x}y'\right) - \lambda e^{-2x}y = 0 \ \Rightarrow\ \frac{d}{dx}\left(e^{-2x}y'\right) + \mu e^{-2x}y = 0$$

(b) This is a regular Sturm–Liouville problem with weight function $p(x) = e^{-2x}$

The orthogonality condition is $\int_0^1 e^{-2x}y_n(x)y_m(x)\ dx = 0$ for $n \neq m$

**Q:7** (16 points ) Find the first two terms of the eigenfunction expansion of $f(x) = \sin(\pi x)$ in eigenfunctions of the Sturm-Liouville problem

$y'' + \lambda y = 0,\ x \in (0, 1)$ with $y'(0) = 0, y'(1) = 0.$

**Sol:** For $\lambda = 0$ the nontrivial solution is $y_0 = c = 1$ (constant solution)

For $\lambda = -\alpha^2,\ \alpha > 0$, the solution is trivial.

For $\lambda = \alpha^2,\ \alpha > 0$, the solution is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$

and $y' = -c_1\alpha \sin \alpha x + c_2\alpha \cos \alpha x$

The condition $y'(0) = 0$ implies $c_2 = 0$ and $y'(1) = 0$ implies $\alpha = n\pi$

The nontrivial solutions are $y_n(x) = \cos n\pi x$

The eigenfunction expansion of $f(x) = \sin(\pi x)$ is

$$\sin(\pi x) = c_0 y_0(x) + c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + \cdots = \sum_{n=0}^{\infty} c_n y_n(x)$$

where $c_n = \dfrac{\int_0^1 \sin(\pi x) y_n(x)\ dx}{\int_0^1 y_n^2(x)\ dx}$

$$c_0 = \frac{\int_0^1 \sin(\pi x)\ dx}{\int_0^1 1\ dx} = \frac{2}{\pi}, \quad c_1 = \frac{\int_0^1 \sin(\pi x)\cos \pi x\ dx}{\int_0^1 \cos^2 \pi x\ dx} = \frac{\frac{1}{2}\int_0^1 \sin 2\pi x\ dx}{\int_0^1 \frac{1+\cos 2\pi x}{2}\ dx} = 0$$

$$c_2 = \frac{\int_0^1 \sin(\pi x)\cos 2\pi x\ dx}{\int_0^1 \cos^2 2\pi x\ dx} = \frac{\frac{1}{2}\int_0^1 \sin 3\pi x - \sin \pi x\ dx}{\int_0^1 \frac{1+\cos 4\pi x}{2}\ dx} = \frac{-\frac{1}{2}\frac{\cos 3\pi x}{3\pi} + \frac{1}{2}\frac{\cos \pi x}{\pi}\Big|_0^1}{\frac{1}{2}}$$

$$= 2\left(\frac{1}{3\pi} - \frac{1}{\pi}\right) = -\frac{4}{3\pi}$$

$$\sin \pi x = \frac{2}{\pi} - \frac{4}{3\pi}\cos 2\pi x + \cdots$$