King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Solution Math 301 Major Exam 2
The Second Semester of 2010-2011 (102)

Q:1 (15 points) Let $\vec{F}(x,y,z) = 15x^2y\mathbf{i} + x^3z^2\mathbf{j} + x^2y^2\mathbf{k}$ and D is the region bounded by the planes $x+y=2,\ z=x+y,\ z=3,\ x=0,\ y=0.$ Use divergence theorem to evaluate $\int\int_S \left(\vec{F}\cdot\hat{n}\right) ds.$

Sol: $\nabla\vec{F} = 30xy$ and $\int\int_S \left(\vec{F}\cdot\hat{n}\right) ds = \int_0^2\int_0^{2-x}\int_{x+y}^{3}(30xy)dzdydx$

$= \int_0^2\int_0^{2-x}(90xy - 30x^2y - 30xy^2)dydx$

$= \int_0^2(100x - 120x^2 + 45x^3 - 5x^4)dx = 28.$

Q:2 (12 points) Use definition to find Laplace transform of $f(t) = e^{-t}\sin(t)$.

Sol: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-t}\sin(t)e^{-st}dt = \int_0^\infty e^{-(s+1)t}\sin(t)dt$

$$= \left[\frac{e^{-(s+1)t}\sin(t)}{-(s+1)}\right]_0^\infty - \frac{1}{-(s+1)}\int_0^\infty e^{-(s+1)t}\cos(t)dt$$

$$= \frac{1}{(s+1)}\int_0^\infty e^{-(s+1)t}\cos(t)dt = \frac{1}{s+1}\left[\frac{e^{-(s+1)t}\cos(t)}{-(s+1)}\right]_0^\infty - \frac{1}{-(s+1)^2}\int_0^\infty e^{-(s+1)t}(-\sin(t))dt$$

$$= \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2}\mathcal{L}\{f(t)\} \Rightarrow \frac{(s+1)^2+1}{(s+1)^2}\mathcal{L}\{f(t)\} = \frac{1}{(s+1)^2} \Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{(s+1)^2+1}.$$

Q:3 (12 points) Use Laplace transform to solve the initial value problem

$y'' + 4y' + 6y = 1 + e^{-t},\ y(0) = 0,\ y'(0) = 0.$

Sol: $s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 6Y(s) = \dfrac{1}{s} + \dfrac{1}{s+1}$

$$(s^2+4s+6)Y(s) = \frac{1}{s} + \frac{1}{s+1} = \frac{2s+1}{s(s+1)}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{\frac{1}{3}}{s+1} + \frac{\frac{1}{6}}{s} - \frac{\frac{1}{2}s+\frac{5}{3}}{s^2+4s+6} \text{ (Using Partial fractions)}$$

$$= \frac{\frac{1}{3}}{s+1} + \frac{\frac{1}{6}}{s} - \frac{\frac{1}{2}s+1+\frac{2}{3}}{s^2+4s+6} = \frac{\frac{1}{3}}{s+1} + \frac{\frac{1}{6}}{s} - \frac{1}{2}\frac{s+2}{s^2+4s+6} - \frac{2}{3}\frac{1}{s^2+4s+6}$$

$$= \frac{\frac{1}{3}}{s+1} + \frac{\frac{1}{6}}{s} - \frac{1}{2}\frac{s+2}{(s+2)^2+2} - \frac{2}{3}\frac{1}{(s+2)^2+2}$$

$$y(t) = \frac{1}{3}e^{-t} + \frac{1}{6} - \frac{1}{2}e^{-2t}\cos(\sqrt{2}\,t) - \frac{2}{3\sqrt{2}}e^{-2t}\sin(\sqrt{2}\,t).$$

Q:4 (12 points) Use Laplace transform to solve to integral equation

$$f(t) = e^{-t} + t^2 - \int_0^t f(\tau)e^{t-\tau}d\tau \text{ for } f(t).$$

Sol: $F(s) = \dfrac{1}{s+1} + \dfrac{2}{s^3} - F(s)\dfrac{1}{s-1}$

$$\frac{s}{s-1}F(s) = \frac{1}{s+1} + \frac{2}{s^3}$$

$$F(s) = \frac{s-1}{s(s+1)} + \frac{2s-2}{s^4} = \frac{1}{s+1} + \frac{-1}{s(s+1)} + \frac{2}{s^3} - \frac{2}{s^4}$$

$$= \frac{1}{s+1} + \frac{-1}{s} + \frac{1}{s+1} + \frac{2}{s^3} - \frac{2}{s^4}$$

$$= \frac{2}{s+1} + \frac{-1}{s} + \frac{2}{s^3} - \frac{2}{s^4}$$

$$f(t) = 2e^{-t} - 1 + t^2 - \frac{1}{3}t^3.$$

Q:5 (12 points) Solve the initial value problem using Laplace transform

$$y'' + 4y' + 5y = \delta(t - 2\pi),\ y(0) = 1,\ y'(0) = 0.$$

Sol: $s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) = e^{-2\pi s}$

$$(s^2 + 4s + 5)Y(s) = s + 4 + e^{-2\pi s}$$

$$Y(s) = \frac{s+4}{(s+2)^2+1} + \frac{e^{-2\pi s}}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1} + \frac{e^{-2\pi s}}{(s+2)^2+1}$$

$$y(t) = e^{-2t}\cos(t) + 2e^{-2t}\sin(t) + e^{-2(t-2\pi)}\sin(t-2\pi)\mathcal{U}(t-2\pi).$$

Q:6 (12 points) Find c_1 and c_2 such that both $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal

to $f_3(x) = 3 + x + c_1x^2 + c_2x^3$ on the interval $[-2, 2]$.

Sol: $(f_1, f_3) = \int_{-2}^{2}(3x + x^2 + c_1x^3 + c_2x^4)dx = \dfrac{16}{3} + \dfrac{64}{5}c_2 = 0 \Rightarrow c_2 = -\dfrac{5}{12}$

$$(f_2, f_3) = \int_{-2}^{2}(3x^2 + x^3 + c_1x^4 + c_2x^5)dx = 16 + \frac{64}{5}c_1 = 0 \Rightarrow c_1 = -\frac{5}{4}$$

Q:7 (15 points) Find Fourier series expansion of $f(x) = x + \pi$, for $-\pi < x < \pi$. Using the result and the fact that given function is continuous at $\frac{\pi}{2}$, show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Sol: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)dx = 2\pi$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos(nx)dx = \frac{1}{\pi}\left[(x+\pi)\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin(nx)dx$

$= \frac{1}{n\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} = 0$, because $(\cos(x) = \cos(-x)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin(nx)dx = \frac{1}{\pi}\left[(x+\pi)\frac{-\cos(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}(-\cos(nx))dx$

$= -2\frac{(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right] = \pi + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n}\sin(nx)\right]$

Since f is continuous at $\frac{\pi}{2}$, therefore, $f(\frac{\pi}{2}) = \frac{\pi}{2} + \pi$

$\frac{\pi}{2} + \pi = \pi + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n}\sin(n\frac{\pi}{2})\right]$

$\frac{\pi}{4} = 1 - 0 - \frac{1}{3} - 0 + \frac{1}{5} - \cdots = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$