King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
SOLUTION Math 301 Major Exam 2
The First Semester of 2010-2011 (101)

Q:1(14 points) Use Laplace transform to solve the initial value problem

$$y'' - y' = \cos(t),\ y(0) = 0,\ y'(0) = -1.$$

Sol: Taking Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) = \frac{s}{s^2+1}$$

$$(s^2 - s)Y(s) = \frac{s}{s^2+1} - 1$$

$$Y(s) = \frac{-s^2+s-1}{(s^2+1)(s^2-s)} = \frac{-s^2+s-1}{(s^2+1)s(s-1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

$$\Longrightarrow\ Y(s) = \frac{1}{s} + \frac{\frac{-1}{2}}{s-1} + \frac{\frac{-1}{2}s + \frac{-1}{2}}{s^2+1}$$

$$\Longrightarrow\ y(t) = 1 - \tfrac{1}{2}e^t - \tfrac{1}{2}\cos(t) - \tfrac{1}{2}\sin(t)$$

Q:2 (a) (6 points) Find Laplace transform $\mathcal{L}\{t\cos(2t)\}$.

(b) (6 points) Find Laplace transform $\mathcal{L}\{e^t\sin(2t)\cos(2t)\}$.

(c) (8 points) Find inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\}$.

Sol: (a) $\mathcal{L}\{t\cos(2t)\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$

(b) $\mathcal{L}\{e^t\sin(2t)\cos(2t)\} = \frac{1}{2}\mathcal{L}\{e^t\sin(4t)\} = \frac{1}{2}\frac{4}{(s-1)^2+16} = \frac{2}{(s-1)^2+16}$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)}\frac{1}{(s^2+9)}\right\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)}\frac{3}{(s^2+9)}\right\}$

$$= \tfrac{1}{9}\sin(3t) \star \sin(3t) = \tfrac{1}{9}\int_0^t \sin(3t-3\tau)\sin(3\tau)d\tau$$

$$= \tfrac{1}{18}\int_0^t \cos(3t-6\tau) - \cos(3t)d\tau = \frac{1}{18}\left[\frac{\sin(3t-6\tau)}{-6} - \tau\cos(3t)\right]_0^t$$

$$= \frac{1}{18}\left[\frac{\sin(3t)}{6} - t\cos(3t) + \frac{\sin(3t)}{6}\right] = \frac{1}{18}\left[\frac{\sin(3t)}{3} - t\cos(3t)\right]$$

Q:3 (a) (8 points) Use Laplace transform to solve the Volterra integral equation

$$f(t) = 3t^2 - \int_0^t f(\tau)e^{t-\tau}d\tau.$$

(b) (8 points) Use Laplace transform to solve the initial value problem

$$y'' + 4y' + 5y = \delta(t - 2\pi), \; y(0) = 0, \; y'(0) = 0.$$

Sol: (a) Taking Laplace transform, we get

$$F(s) = \frac{6}{s^3} - \frac{1}{s-1}F(s) \implies \left(1 + \frac{1}{s-1}\right)F(s) = \frac{6}{s^3} \implies \left(\frac{s}{s-1}\right)F(s) = \frac{6}{s^3}$$

$$\implies F(s) = \frac{6(s-1)}{s^4} = \frac{6}{s^3} - \frac{6}{s^4} \implies f(t) = 3t^2 - t^3$$

(b) Taking Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) = e^{-2\pi s}$$

$$(s^2 + 4s + 5)Y(s) = e^{-2\pi s} \implies Y(s) = \frac{e^{-\pi s}}{s^2 + 4s + 5} = \frac{e^{-\pi s}}{(s+2)^2 + 1}$$

$$\implies y(t) = e^{-2(t-2\pi)}\sin(t - 2\pi)u(t - 2\pi) = e^{-2t+4\pi}\sin(t)u(t - 2\pi)$$

Q:4 (14 points) Show that the set of functions $\{\cos(2n+1)x\}$ is an orthogonal set on $\left[0, \frac{\pi}{2}\right]$ for $n = 0, 1, 2, 3, \cdots$. Also find norm of each function. (Justify your answer with reason).

Sol: Let $\phi_n(x) = \cos(2n+1)x$. Then for $n \neq m$

$$(\phi_n, \phi_m) = \int_0^{\frac{\pi}{2}} \cos(2n+1)x.\cos(2m+1)x dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\cos(2n+2m+2)x + \cos(2n-2m)x\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{(2n+2m+2)}\sin(2n+2m+2)x + \frac{1}{(2n-2m)}\sin(2n-2m)x\right]_0^{\frac{\pi}{2}} = 0.$$

$$\|\phi_n\|^2 = \int_0^{\frac{\pi}{2}} \cos^2(2n+1)x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1 + \cos(4n+2)x dx = \frac{1}{2}\left[x + \frac{1}{4n+2}\sin((4n+2)x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

So $\|\phi_n\| = \frac{\sqrt{\pi}}{2}$

Q:5 (14 points) Find Fourier series expansion of

$$f(x) = \begin{cases} 0 & \text{if } -3 < x < 0 \\ 3 - x & \text{if } 0 \le x < 3 \end{cases} .$$

Sol: $a_0 = \frac{1}{3}\int_0^3 (3-x)dx = \frac{1}{3}\left[3x - \frac{x}{2}\right]_0^3 = \frac{3}{2}$

$$a_n = \frac{1}{3}\int_0^3 (3-x)\cos(\tfrac{n\pi}{3})xdx = \frac{1}{3}\left[\tfrac{3}{n\pi}(3-x)\sin(\tfrac{n\pi}{3})x\right]_0^3 + \frac{1}{3}\int_0^3 \tfrac{3}{n\pi}\sin(\tfrac{n\pi}{3})xdx$$

$$= \tfrac{-3}{n^2\pi^2}\left[\cos(\tfrac{n\pi}{3})x\right]_0^3 = \frac{3(1-(-1)^n)}{n^2\pi^2}$$

$$b_n = \frac{1}{3}\int_0^3 (3-x)\sin(\tfrac{n\pi}{3})xdx = \frac{1}{3}\left[\tfrac{-3}{n\pi}(3-x)\cos(\tfrac{n\pi}{3})x\right]_0^3 - \frac{1}{3}\int_0^3 \tfrac{3}{n\pi}\cos(\tfrac{n\pi}{3})xdx$$

$$= \frac{1}{3}\left[0 + \tfrac{9}{n\pi}\right] - 0 = \frac{3}{n\pi}$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{3(1-(-1)^n)}{n^2\pi^2}\cos(\tfrac{n\pi}{3})x + \frac{3}{n\pi}\sin(\tfrac{n\pi}{3})x\right]$$

Q:6 (10 points) Find half range cosine expansion of

$$f(x) = x + 1, \quad 0 < x < \pi.$$

Sol: $a_0 = \frac{2}{\pi}\int_0^\pi (x+1)dx = \frac{2}{\pi}\left[\frac{x^2}{2} + x\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^2}{2} + \pi\right] = \pi + 2$

$$a_n = \frac{2}{\pi}\int_0^\pi (x+1)\cos(\tfrac{n\pi}{\pi})xdx = \frac{2}{\pi}\left[\tfrac{1}{n}(x+1)\sin(nx)\right]_0^\pi - \frac{2}{\pi}\int_0^\pi \tfrac{1}{n}\sin(nx)dx$$

$$= \tfrac{2}{n^2\pi}\left[\cos(nx)\right]_0^\pi = \frac{2((-1)^n - 1)}{n^2\pi}$$

$$f(x) = \frac{\pi+2}{2} + \sum_{n=1}^{\infty}\left[\frac{2((-1)^n-1)}{n^2\pi}\cos(nx)\right]$$