King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Major Exam 2
The Summer Semester of 2009-2010 (093)

Q:1 (a) (5 points) Use Laplace transform to solve the initial value problem

$$y'' - 2y' + 5y = 0,\ y(0) = 1,\ y'(0) = 3.$$

(b) (5 points) Find Laplace transform of $f(t) = \begin{cases} 0 & \text{if } 0 \le t < \frac{\pi}{2} \\ \sin t & \text{if } t \ge \frac{\pi}{2} \end{cases}$.

Sol.(a) $\mathcal{L}\{y'' - 2y' + 5y\} = 0 \Rightarrow s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = 0$

$$\left(s^2 - 2s + 5\right) Y(s) = s + 3 - 2 = s + 1$$

$$Y(s) = \frac{s+1}{s^2 - 2s + 5} = \frac{s+1}{(s-1)^2 + 2^2} = \frac{s-1}{(s-1)^2 + 2^2} + \frac{2}{(s-1)^2 + 2^2}$$

$$y(t) = e^t \cos(2t) + 2e^t \sin(2t)$$

(b) $f(t) = \sin(t)\, u\left(t - \frac{\pi}{2}\right) = \cos\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\cos\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right)\right\} = \frac{s}{s^2+1} e^{-\frac{\pi}{2}s}.$$

Q:2 (a) (5 points) Find Laplace transform $\mathcal{L}\{te^{-3t}\cos^2 t\}$.

(b) (5 points) Find Laplace transform $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-2)}\right\}$.

(c) (6 points) Use Laplace transform to solve the initial value problem

$$y'' + y = \delta(t - 2\pi) + \delta(t - 3\pi),\ y(0) = 1,\ y'(0) = 0.$$

Sol. (a) $\mathcal{L}\{te^{-3t}\cos^2 t\} = \mathcal{L}\left\{te^{-3t}\left(\frac{1+\cos 2t}{2}\right)\right\} = \frac{1}{2}\mathcal{L}\{te^{-3t}\} + \mathcal{L}\{te^{-3t}\cos 2t\}$

$$= \frac{1}{2}\frac{1}{(s+3)^2} - \frac{d}{ds}\left(\frac{s+3}{(s+3)^2+4}\right) = \frac{1}{2}\frac{1}{(s+3)^2} - \left(\frac{(s+3)^2 + 4 - (s+3)(2s+6)}{\left((s+3)^2+4\right)^2}\right)$$

$$= \frac{1}{2}\frac{1}{(s+3)^2} + \frac{s^2+6s+5}{\left((s+3)^2+4\right)^2}$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s-2)}\right\} = \dfrac{1}{4}e^{2t} - \dfrac{1}{2}t - \dfrac{1}{4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{4s} + \frac{-1}{2s^2} + \frac{1}{4(s-2)}\right\} = -\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}.$$

(c) $\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t-2\pi) + \delta(t-3\pi)\}$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = e^{-2\pi s} + e^{-3\pi s}$$

$$(s^2+1)\, Y(s) = s + e^{-2\pi s} + e^{-3\pi s}$$

$$Y(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}\left(e^{-2\pi s} + e^{-3\pi s}\right)$$

$$y(t) = \cos t + \sin(t-2\pi)\, u(t-2\pi) + \sin(t-3\pi)\, u(t-3\pi)$$

Q:3 (10 points) Show that the set of functions $\left\{1,\ \cos\left(\frac{n\pi}{p}\right)x\right\}$ is an orthogonal set on $[0,p]$ for $n = 1, 2, 3, \cdots$. Also find norm of each function. (Justify your answer with reason).

Sol. Let $\phi_0 = 1$ and $\phi_n = \cos\left(\frac{n\pi}{p}x\right)$

$$(\phi_0, \phi_n) = \int_0^p \cos\left(\tfrac{n\pi}{p}\right)x\, dx = \frac{p}{n\pi}\sin\left(\tfrac{n\pi}{p}\right)x\Big|_0^p = \frac{p}{n\pi}\left(\sin(n\pi) - \sin(0)\right) = 0$$

$$(\phi_n, \phi_m) = \int_0^p \cos\left(\tfrac{n\pi}{p}x\right)\cos\left(\tfrac{m\pi}{p}x\right)dx = \frac{1}{2}\int_0^p \left[\cos\left(\tfrac{(n+m)\pi}{p}x\right) + \cos\left(\tfrac{(n-m)\pi}{p}x\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{p}{(n+m)\pi}\sin\left(\tfrac{(n+m)\pi}{p}x\right) + \frac{p}{(n-m)\pi}\sin\left(\tfrac{(n-m)\pi}{p}x\right)\right]\Bigg|_0^p = 0$$

$$\|\phi_0\|^2 = (\phi_0, \phi_0) = \int_0^p dx = p. \text{ So } \|\phi_0\| = \sqrt{p}.$$

$$\|\phi n\|^2 = (\phi_n, \phi_n) = \int_0^p \cos^2\left(\tfrac{n\pi}{p}x\right)dx = \frac{1}{2}\int_0^p \left(1 + \cos\left(\tfrac{2n\pi}{p}x\right)\right)dx$$

$$= \frac{1}{2}\left(x + \frac{p}{2n\pi}\sin\left(\tfrac{2n\pi}{p}x\right)\right)\Big|_0^p = \frac{p}{2}$$

So $\|\phi n\| = \sqrt{\dfrac{p}{2}}$.

Q:4 (12 points) Find Fourier series expansion of $f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ x & \text{if } 0 \le x < 1 \end{cases}$.

Sol. $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1dx + \int_{0}^{1} xdx = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{0}^{1} x\cos(n\pi x)\,dx$$

$$= 0 + \left(\frac{x\sin(n\pi x)}{n\pi}\right)\Bigg|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}dx = 0 + \frac{1}{n^2\pi^2}\left(\cos(n\pi x)\right)\big|_0^1 = \frac{(-1)^n - 1}{n^2\pi^2}.$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} \sin(n\pi x)\,dx + \int_{0}^{1} x\sin(n\pi x)\,dx$$

$$= \frac{-\cos(n\pi x)}{n\pi}\Bigg|_{-1}^{0} + \left(\frac{-x\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}dx$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{(-1)^n}{n\pi} + 0 = \frac{-1}{n\pi}.$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)\right]$$

Q:5 (a) (10 points) Find eigenvalues and eigenfunctions of the boundary value problem

$$x^2y'' + xy' + \lambda y = 0,\ \ y(1) = 0,\ \ y(5) = 0.$$

(b) (3 points) Write the given differential equation into self-adjoint form.

(c) (2 points) Write the orthogonality condition.

Sol. The auxiliary equation is $m(m-1) + m + \lambda = 0 \ \Rightarrow m^2 + \lambda = 0$

Case I: $\lambda = 0$, then $m = 0, 0$ and $y = c_1 + c_2 \ln(x)$

$y(1) = 0 \ \Rightarrow c_1 = 0$ and $y(5) = 0 \ \Rightarrow c_2 \ln(5) = 0 \ \Rightarrow c_2 = 0$. Trivial solution.

Case II: $\lambda = -\alpha^2,\ \alpha > 0$, then $m = \pm\alpha$ and $y = c_1\cosh(\alpha x) + c_2\sinh(\alpha x)$

$y(1) = 0 \ \Rightarrow c_1 = 0$ and $y(5) = 0 \ \Rightarrow c_2\sinh(5\alpha) = 0 \ \Rightarrow c_2 = 0.$

Trivial solution because $\sinh(5\alpha) > 0$ for $\alpha > 0$.

Case III: $\lambda = \alpha^2,\ \alpha > 0$, then $m = \pm\alpha i$ and $y = c_1\cos(\alpha x) + c_2\sin(\alpha x)$

$y(1) = 0 \ \Rightarrow c_1 = 0$ and $y(5) = 0 \ \Rightarrow c_2\sin(5\alpha) = 0$

For nontrivial solution $c_2 \neq 0$ and $\sin(5\alpha) = 0 \ \Rightarrow \alpha_n = \dfrac{n\pi}{5},\ n = 1, 2, 3, ...$

$y_n = \sin\left(\dfrac{n\pi}{5}x\right)$ are the eigenfunctions corresponding to eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{25},\ n = 1, 2, ...$

Q:6 (12 points) Solve the Heat equation using separation constant $\lambda = \alpha^2,\ \alpha > 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

subject to the conditions $u(0,t) = 0,\ u(L,t) = 0,\ u(x,0) = \begin{cases} 1 & \text{if } 0 < x < \dfrac{L}{2} \\ 0 & \text{if } \dfrac{L}{2} < x < L \end{cases}$.

Sol. Let $u(x,t) = X(x)T(t)$, then $\dfrac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\dfrac{\partial u}{\partial t} = X(x)T'(t)$.

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} \Rightarrow X''(x)T(t) = X(x)T'(t) \Rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{T(t)} = -\lambda$

$X''(x) + \lambda X(x) = 0$ and $T'(t) + \lambda T(t) = 0$.

For $\lambda = \alpha^2$, the solutions are $X(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$ and $T(t) = c_3 e^{-\alpha t}$

$u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow c_1 = 0$ and $u(L,t) = 0 \Rightarrow X(L) = 0 \Rightarrow c_2\sin(\alpha L) = 0$

For non trivial solution $c_2 \neq 0$, so $\sin(\alpha L) = 0 \Rightarrow \alpha_n = \dfrac{n\pi}{L}$ for $n = 1, 2, 3, ...$

So $u_n(x,t) = c_2 c_3 \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t},\ n = 1, 2, 3, ...$

By super position principle $u(x,t) = \displaystyle\sum_{n=0}^{\infty} A_n \sin\left(\tfrac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$

Now $u(x,0) = f(x) \ \Rightarrow f(x) = \displaystyle\sum_{n=0}^{\infty} A_n \sin\left(\tfrac{n\pi}{L}x\right)$, where $A_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\tfrac{n\pi}{L}x\right)dx$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\tfrac{n\pi}{L}x\right)dx = A_n = \frac{2}{L}\int_0^{\frac{L}{2}} \sin\left(\tfrac{n\pi}{L}x\right)dx = -\frac{2}{L}\frac{L}{n\pi}\left(\cos\left(\tfrac{n\pi}{L}x\right)\right)\Big|_0^{\frac{L}{2}}$$

$$= -\frac{2}{n\pi}\left(\cos\left(\tfrac{n\pi}{2} - 1\right)\right)$$

$$u(x,t) = \sum_{n=0}^{\infty} -\frac{2}{n\pi}\left(\cos\left(\tfrac{n\pi}{2} - 1\right)\right)\sin\left(\tfrac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}t}$$