King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
SOLUTION Math 301 Major Exam 1
The Third Semester of 2012-2013 (123)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Instructor: ______________________ Sec #: ________ Serial #: ________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		10
2		12
3		10
4		12
5		12
6		14
7		15
8		15
Total		100

Q:1 (10 points) Find length of the curve traced by $\vec{r}(t) = e^{2t}\cos(3t)\,\mathbf{i} + e^{2t}\sin(3t)\mathbf{j} + e^{2t}\mathbf{k}$ on the interval $0 \le t \le 2\pi$. Also find a unit tangent vector to the curve at $t = \pi$.

Sol: $\vec{r}\,'(t) = (2e^{2t}\cos(3t) - 3e^{2t}\sin(3t))\,\mathbf{i} + (2e^{2t}\sin(3t) + 3e^{2t}\cos(3t))\,\mathbf{j} + 2e^{2t}\,\mathbf{k}$

$$\|\vec{r}\,'(t)\|^2 = 4e^{4t}\cos^2(3t) - 6e^{4t}\cos(3t)\sin(3t) + 9e^{4t}\sin^2(3t) + 4e^{4t}\sin^2(3t) + 6e^{4t}\sin(3t)\cos(3t)$$
$$+9e^{4t}\cos^2(3t) + 4e^{4t} = 17e^{4t}$$

Arc Length $S = \int_0^{2\pi} \|\vec{r}\,'(t)\| dt = \int_0^{2\pi} \sqrt{17}\, e^{2t} dt = \frac{\sqrt{17}}{2}(e^{4\pi} - 1)$

$\vec{r}\,'(\pi) = (-2e^{2\pi})\,\mathbf{i} + (-3e^{2\pi})\,\mathbf{j} + 2e^{2\pi}\,\mathbf{k}$

Unit vector is $\dfrac{(-2e^{2\pi})\,\mathbf{i} + (-3e^{2\pi})\,\mathbf{j} + 2e^{2\pi}\,\mathbf{k}}{\sqrt{17}e^{2\pi}} = \dfrac{-2\,\mathbf{i} - 3\,\mathbf{j} + 2\,\mathbf{k}}{\sqrt{17}}$

Q:2 (12 points) Let $f(x, y) = 2xy - x^2y$.

(a) Find the directional derivative of $f(x, y)$ at $(3, 2)$ in the direction of a tangent vector to the graph of $x^2 + 2y^2 = 6$ at $(2, 1)$.

(b) Find the direction and value of maximum directional derivative of $f(x, y)$ at $(3, 2)$.

Sol:(a) $\nabla f(x, y) = (2y - 2xy)\mathbf{i} + (2x - x^2)\mathbf{j}$

$\nabla f(3, 2) = -8\mathbf{i} - 3\mathbf{j}$

Now $2x + 4yy' = 0 \;\Rightarrow y' = -\dfrac{2x}{4y} = -\dfrac{x}{2y} \;\Rightarrow\; y' = -1$ at $(2, 1)$

Then the tangent line to $x^2 + 2y^2 = 6$ at $(2, 1)$ is $y - 1 = -1(x - 2) \;\Rightarrow\; \dfrac{x-2}{1} = \dfrac{y-1}{-1}$

A unit vector in the direction of this tangent line is $\hat{u} = \pm\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$

So $D_{\hat{u}}f(3, 2) = \pm\frac{-5}{\sqrt{2}}$

(b) Direction of maximum directional derivative is $-8\mathbf{i} - 3\mathbf{j}$ and value of maximum directional derivative is $\sqrt{73}$.

Q:3 (10 points) Let $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a constant vector. Show that

(a) $\nabla \times [(\vec{r} \cdot \vec{r})\, \vec{a}] = 2(\vec{r} \times \vec{a})$

(b) $\nabla \cdot [(\vec{r} \cdot \vec{r})\, \vec{a}] = 2(\vec{r} \cdot \vec{a})$

Sol:(a) $(\vec{r} \cdot \vec{r})\, \vec{a} = (x^2 + y^2 + z^2)(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$

$$\nabla \times [(\vec{r} \cdot \vec{r})\, \vec{a}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2+z^2)a_1 & (x^2+y^2+z^2)a_2 & (x^2+y^2+z^2)a_3 \end{vmatrix}$$

$$= (2ya_3 - 2za_2)\mathbf{i} + (2za_1 - 2xa_3)\mathbf{j} + (2xa_2 - 2ya_1)\mathbf{k} = 2(\vec{r} \times \vec{a})$$

(b) $\nabla \cdot [(\vec{r} \cdot \vec{r})\, \vec{a}] = 2xa_1 + 2ya_2 + 2za_3 = 2(xa_1 + ya_2 + za_3) = 2(\vec{r} \cdot \vec{a})$

Q:4 (12 points) Find work done by the force $\vec{F} = (2xy - e^{3y})\mathbf{i} + (x^2 - 3xe^{3y})\mathbf{j}$ along the curve $y = x^4$ for $0 \le x \le 1$.

Sol: Since $\dfrac{\partial P}{\partial y} = 2x - 3e^{3y} = \dfrac{\partial Q}{\partial x}$, therefore $\overrightarrow{F}$ is conservative and we can choose any path from $(0,0)$ to $(1,1)$

Let $y = x$ be the new path from $(0,0)$ to $(1,1)$.

Then W$=\int_C \vec{F} \cdot d\vec{r} = \int_C (2xy - e^{3y})dx + (x^2 - 3xe^{3y})dy = \int_0^1 (2x^2 - e^{3x})dx + (x^2 - 3xe^{3x})dx$

$$= \int_0^1 (3x^2 - (1+3x)e^{3x})dx = 1 - e^3.$$

Q:5 (12 points) Use Green's theorem to evaluate the line integral

$\oint_C (2x^2 - 2y^3)dx + (2x^3 + 3y^2)dy,$

$C = C_1 \bigcup C_2$, where C_1 is a positively oriented circle $x^2 + y^2 = 9$ and C_2 is a negatively oriented circle $x^2 + y^2 = 4$.

Sol: $\oint_C (2x^2 - 2y^3)dx + (2x^3 + 3y^2)dy = \iint_R (6x^2 + 6y^2)dA = \int_0^{2\pi}\int_2^3 6r^3 dr d\theta = 3\pi(r^4)|_2^3 = 195\pi.$

Q:6 (14 points) Evaluate the surface integral $\iint_S (2xz + 3yz)\, dS$, where S is the portion of the plane $3x + 2y + 4z = 12$ in the first octant. Use projection of S onto yz-plane.

Sol: $x = 4 - \frac{2}{3}y - \frac{4}{3}z$ and $dS = \sqrt{1 + \frac{4}{9} + \frac{16}{9}}dA = \frac{\sqrt{29}}{3}dA$

$$\iint_S (2xz + 3yz)\, dS = \iint_R (2xz + 3yz)\frac{\sqrt{29}}{3}\, dA = \frac{\sqrt{29}}{3}\iint_R (2z(4 - \tfrac{2}{3}y - \tfrac{4}{3}z) + 3yz)\, dA$$

$$= \frac{\sqrt{29}}{3}\int_0^3\int_0^{6-2z} (8z + \tfrac{5}{3}yz - \tfrac{8}{3}z^2)\, dydz = \frac{\sqrt{29}}{3}\int_0^3 (8z(6-2z) + \tfrac{5}{6}z(6-2z)^2 - \tfrac{8}{3}z^2(6-2z)dz$$

$$= \frac{\sqrt{29}}{3}\int_0^3 (78z - 52z^2 + \tfrac{26}{3}z^3)dz = \frac{117\sqrt{29}}{6}.$$

Q:7 (15 points) Use Stokes' theorem to evaluate the integral $\iint\limits_S curl(\vec{F}) \cdot \hat{n}\ dS$, where

$\vec{F} = \dfrac{xz}{4}\mathbf{i} + 4xy\mathbf{j} + 3xyz\mathbf{k}$ and S is the portion of the paraboloid $z = x^2 + 4y^2$

for $0 \leq z \leq 16$.

Sol: $\iint\limits_S curl(\vec{F}) \cdot \hat{n}\ dS = \oint\limits_C \vec{F} \cdot d\vec{r}$

The curve C is $x^2 + 4y^2 = 16$ whose parametric equations are

$x = 4\cos(t),\ y = 2\sin(t),\ 0 \leq t \leq 2\pi$

So $\oint\limits_C \vec{F} \cdot d\vec{r} = \int\limits_0^{2\pi} [(-64\cos(t)\sin(t)) + 64\sin(t)\cos^2(t)]\ dt = 0.$

Q:8 (15 points) Use divergence theorem to evaluate $\iint\limits_S (\vec{F}.\hat{n})\ dS$ where $\vec{F} = 6xz\mathbf{i} + 5y^2\mathbf{j} - 3z^2\mathbf{k}$

and D the region bounded by $z = y$, $z = 4 - y$, $z = 2 - \frac{1}{2}x^2$, $x = 0$ and $z = 0$.

Sol: $div(\vec{F}) = 6z + 10y - 6z = 10y$

$$\iint\limits_S (\vec{F}.\hat{n})\ dS = \iiint\limits_D div(\vec{F})dV = \int\limits_0^2 \int\limits_0^{2-\frac{1}{2}x^2} \int\limits_z^{4-z} (10y)dydzdx$$

$$= \int\limits_0^2 \int\limits_0^{2-\frac{1}{2}x^2} (5(4-z)^2 - 5z^2)dzdx = 5\int\limits_0^2 \int\limits_0^{2-\frac{1}{2}x^2} (16 - 8z)dzdx$$

$$= 5\int\limits_0^2 (16 - 4x^2)dx = 128.$$