King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Solution Math 301 Major Exam 1
The First Semester of 2010-2011 (102)

Q:1 (10 points) Express the vector equation $\vec{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j} + \mathbf{k}$ as function of arc length s. Verify that $\vec{r}\,'(s)$ is a unit tangent vector.

Sol. The arc length function is:

$s = \int_0^t \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 0^2} dt = \int_0^t 3dt = 3t \Rightarrow t = \frac{s}{3}$

The given vector as a function of arc length s is $\vec{r}(s) = 3\cos(\frac{s}{3})\ \mathbf{i} + 3\sin(\frac{s}{3})\ \mathbf{j} + \mathbf{k}$

and $\vec{r}\,'(s) = -3\frac{1}{3}\sin(\frac{s}{3})\ \mathbf{i} + 3\frac{1}{3}\cos(\frac{s}{3})\ \mathbf{j} + 0\ \ \mathbf{k}$ with $\|\vec{r}\,'(s)\| = 1$.

Q:2 (10 points) Find a vector that gives the direction in which the function $f(x,y,z) = xyz + \ln|yz|$ decreases most rapidly at $(1,2,3)$. Find the minimum rate.

Sol. $\nabla f = < yz,\ \ xz + \frac{z}{yz},\ \ xz + \frac{y}{yz} > = < yz,\ \ xz + \frac{1}{y},\ \ xz + \frac{1}{z} >$

and $\nabla f(1,2,3) = \left\langle 6, \frac{7}{2}, \frac{7}{3} \right\rangle$.

The function f decreases most rapidly in the direction of $\left\langle -6, -\frac{7}{2}, -\frac{7}{3} \right\rangle$

The minimum rate is $-||\nabla f(1,2,3)|| = -\sqrt{36 + \frac{49}{4} + \frac{49}{9}} = -\frac{\sqrt{1933}}{4}$

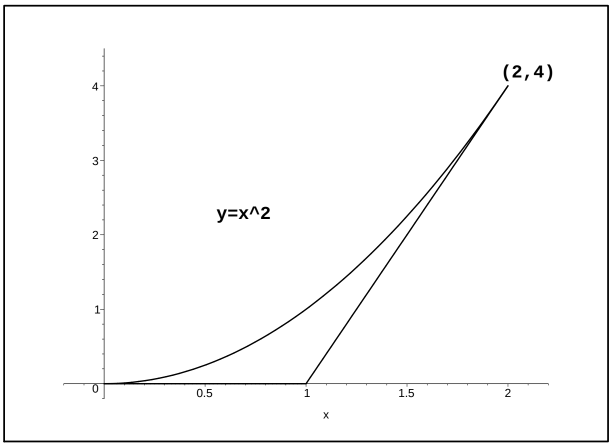

Figure 1: Figure for Q: 3

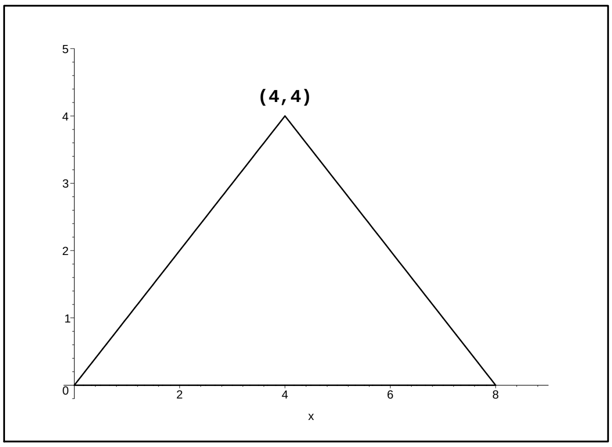

Figure 2: Figure for Q: 4

Q:3 (15 points) Evaluate the line integral $\int_C (x+y)dx + ydy$ along the path shown in the figure:

Sol. The curves in the figure are:

$$C_1 : 0 \le x \le 1,\quad y = 0 \text{ and } C_2 : y = 4x - 4,\quad 1 \le x \le 2 \text{ and } C_3 : y = x^2,\quad 2 \ge x \ge 0$$

$$\int_C (x+y)dx + ydy = \int_{C_1} xdx \;+\; \int_{C_2} (x + 4x - 4)dx + (4x-4)4dx \;+\; \int_{C_3} (x + x^2)dx + x^2 2xdx$$

$$= \int_0^2 xdx \;+\; \int_1^2 (21x - 20)dx \;+\; \int_2^0 (2x^3 + x^2 + x)dx = \frac{1}{2} + \frac{23}{2} - \frac{38}{3} = -\frac{2}{3}$$

Q:4 (12 points) Use Green's theorem to evaluate the line integral $\oint_C y^2dx - x^2ydy$ along the closed path C given in the figure:

Sol. $\oint_C y^2dx - x^2ydy = \int\int_R (-2xy - 2y)dA = -2\int_0^4 \int_y^{8-y} (xy + y)dxdy$

$$-2\int_0^4 y\left(\frac{x^2}{2} + x\right)\Big|_y^{8-y} dy = 20\int_0^4 \left(y^2 - 4y\right)dy = \frac{-640}{3}$$

$x = y$ is the line through $(0,0)$ to $(4,4)$ and $x = 8 - y$ is the line through $(8,0)$ and $(4,4)$.

Q:5 (12 points) Determine if the vector field $\vec{F}(x,y) = y\cos x\ \mathbf{i} + (\sin x + 1)\ \mathbf{j}$ is a conserved field. If so, find a potential function $\phi(x,y)$ for $\vec{F}$. Show that $\vec{F} = grad\phi$.

Sol. $P(x,y) = y\cos x,\ Q(x,y) = \sin x + 1$ and $\dfrac{\partial Q}{\partial x} = \cos x = \dfrac{\partial P}{\partial y}$. So the vector is conserved.

There exists a function $\phi(x,y)$ such that $\nabla\phi = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$

$\dfrac{\partial \phi}{\partial x} = P(x,y) = y\cos(x) \Rightarrow \phi(x,y) = y\sin(x) + g(y)$ and $\dfrac{\partial \phi}{\partial y} = \sin(x) + g'(y) = \sin(x) + 1$

$\Rightarrow g'(y) = 1 \Rightarrow g(y) = y + C$ So $\phi(x,y) = y\sin(x) + y + C$

and $\nabla\phi(x,y) = y\cos x\ \mathbf{i} + (\sin x + 1)\ \mathbf{j} = \vec{F}(x,y)$

Q:6 (16 points) Use Stokes' theorem to evaluate the integral $\oint_C \vec{F}\cdot dr$, where $\vec{F} = -y^3\ \mathbf{i} + x^3\ \mathbf{j} + \cos z\ \mathbf{k}$ and C is the trace of the cylinder $x^2 + y^2 = 4$ in the plane $x + y + 2z = 4$.

Sol. $\nabla\times\vec{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & \cos(z) \end{bmatrix} = 3(x^2+y^2)\mathbf{k}$

Unit normal to the surface, which is portion of the plane $x + y + 2z = 4$ inside the cylinder $x^2 + y^2 = 4$, is $\hat{n} = \dfrac{\mathbf{i}+\mathbf{j}+2\mathbf{k}}{\sqrt{1+1+4}} = \dfrac{\mathbf{i}+\mathbf{j}+2\mathbf{k}}{\sqrt{6}}$ and $(\nabla\times\vec{F})\cdot\hat{n} = \sqrt{6}(x^2+y^2)$

Also for $z = 2 - x/2 - y/2$, $ds = \sqrt{1 + 1/4 + 1/4} = \sqrt{6/4} = \dfrac{\sqrt{6}}{2}$

$\oint_C \vec{F}\cdot dr = \int\int_S (\nabla\times\vec{F})\cdot\hat{n}ds = \sqrt{6}\int\int_R (x^2+y^2)\frac{\sqrt{6}}{2}dA = 3\int_0^{2\pi}\int_0^2 r^2 r dr d\theta = 24\pi.$

Q:7 (15 points) Evaluate the surface integral $\int\int_S G(x,y,z)dS$, where $G(x,y,z) = (x^2+y^2)z$ and S that portion of the sphere $x^2 + y^2 + z^2 = 16$ in the first octant.

Sol. The surface is $z = \sqrt{16 - x^2 - y^2}$ and $\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{16-x^2-y^2}}, \dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{16-x^2-y^2}}$

So $dS = \sqrt{1 + \dfrac{x^2+y^2}{16-x^2-y^2}}dA = \dfrac{4}{\sqrt{16-x^2-y^2}}dA$

$\int\int_S G(x,y,z)dS = \int\int_R (x^2+y^2)\sqrt{16-x^2-y^2}\dfrac{4}{\sqrt{16-x^2-y^2}}dA$

$= 4\int_0^{\pi/2}\int_0^4 r^2 r dr d\theta = 128\pi.$