King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Major Exam I
The Summer Semester of 2009-2010 (093)

Q:1 (a) (7 points) Find the directional derivative of $f(x,y,z) = xy^2 - 4x^2y + z^2$ at $(1,-1,2)$ in the direction of $3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

(b) (5 points) Write the direction of maximum directional derivative and value of maximum directional derivative.

Sol. (a) $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle y^2 - 8xy, 2xy - 4x^2, 2z \rangle$

$\vec{u} = \langle 3, 6, 2 \rangle$, $|\vec{u}| = \sqrt{9 + 36 + 4} = 7$, $\hat{u} = \left\langle \frac{3}{7}, \frac{6}{7}, \frac{2}{7} \right\rangle$

$D_{\hat{u}} f(1,-1,2) = \nabla f(1,-1,2) \cdot \hat{u} = \frac{3}{7}(1+8) + \frac{6}{7}(-2-4) + \frac{2}{7}(4) = -\frac{1}{7}.$

(b) $\nabla f(1,-1,2) = \langle 9, -6, 4 \rangle$ is the direction of maximum directional derivative and value of maximum directional derivative is $\|\nabla f(1,-1,2)\| = \sqrt{81 + 36 + 16} = \sqrt{133}$.

Q:2 Let $\vec{F}(x,y,z) = xye^z\ \mathbf{i} + yze^x\ \mathbf{j} + xze^y\ \mathbf{k}$.

(a) (5 points) Find $\nabla \cdot \vec{F}$.

(b) (5 points) Find $\nabla \times \vec{F}$.

(c) (4 points) Find $\nabla \cdot (\nabla \times \vec{F})$.

Sol. (a) $\nabla \cdot \vec{F} = \frac{\partial xye^z}{\partial x} + \frac{\partial yze^x}{\partial y} + \frac{\partial xze^y}{\partial z} = xe^y + ze^x + ye^z.$

(b)
$$\nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xye^z & yze^x & xze^y \end{vmatrix}$$
$$= \mathbf{i}\left(\frac{\partial xze^y}{\partial y} - \frac{\partial yze^x}{\partial z}\right) + \mathbf{j}\left(\frac{\partial xye^z}{\partial z} - \frac{\partial xze^y}{\partial x}\right) + \mathbf{k}\left(\frac{\partial yze^x}{\partial x} - \frac{\partial xye^z}{\partial y}\right)$$
$$= \mathbf{i}(xze^y - ye^x) + \mathbf{j}(xye^z - ze^y) + \mathbf{k}(yze^x - xe^z)$$

(c)
$$\nabla \cdot \left(\nabla \times \vec{F}\right) = \frac{\partial (xze^y - ye^x)}{\partial x} + \frac{\partial (xye^z - ze^y)}{\partial y} + \frac{\partial (yze^x - xe^z)}{\partial z} = 0.$$

Q:3 (10 points) Determine whether the integral $\int\limits_{(1,2)}^{(3,6)} (3y^2x^2+5)\,dx + (2x^3y-4)\,dy$ is independent of path. If so, use a convenient path between the points and evaluate the integral.

Sol. Let $P(x,y) = 3y^2x^2+5,\ Q(x,y) = 2x^3y-4$. Then $\dfrac{\partial Q}{\partial x} = 6x^2y = \dfrac{\partial P}{\partial y}$. So the integral is independent of path.

Consider the stariaght line path between two points $(1,2)$ and $(3,6)$ which is $y = 2x$.

$$\int\limits_{(1,2)}^{(3,6)} (3y^2x^2+5)\,dx + (2x^3y-4)\,dy = \int\limits_{1}^{3} (12x^4+5)\,dx + (4x^4-4)\,2dx$$

$$= \int\limits_{1}^{3} (20x^4-3)\,dx = 962.$$

Q:4 (12 points) Use Green's Theorem to evaluate the integral $\oint\limits_C (3x+2y^2)\,dx + (3x^2-2y)\,dy$, where C is the boundary of the region determined by the graphs of $y = x^2$ and $y = 1$.

Sol. $\oint\limits_C (3x+2y^2)\,dx + (3x^2-2y)\,dy = \iint\limits_R (6x-4y)\,dA = \int\limits_{-1}^{1}\int\limits_{x^2}^{1} (6x-4y)\,dydx = -\dfrac{16}{5}.$

Q:5 (12 points) Find the surface area of the portions of the sphere $x^2+y^2+z^2 = 16$ that are within the cylinder $x^2+y^2 = 4y$.

Sol. $z = \sqrt{16-x^2-y^2},\ \dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{16-x^2-y^2}},\ \dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{16-x^2-y^2}}$

$$ds = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \sqrt{1+\frac{x^2}{16-x^2-y^2}+\frac{y^2}{16-x^2-y^2}}\,dA = \frac{4}{\sqrt{16-x^2-y^2}}dA$$

Area of upper side is

$$A_1(S) = \iint\limits_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint\limits_R \frac{4}{\sqrt{16-x^2-y^2}}dA$$

$$= \int\limits_0^{\pi}\int\limits_0^{4\sin\theta} 4\left(16-r^2\right)^{-\frac{1}{2}} r\,dr\,d\theta = \int_0^{\pi}\left(16-4\sqrt{8\cos 2\theta+8}\right)\,d\theta = 16\pi-32$$

Total Area $= 2A_1(S) = 32(\pi-2)$.

Q:6 (15 points) Let $\vec{F}(x,y,z) = z\ \mathbf{i} + x\ \mathbf{j} + y\ \mathbf{k}$. Use Stokes' theorem to evaluate the integral $\oint\limits_C F \cdot dr$, where C is the counter clockwise boundary of the surface that is bounded by the plane $2x + y + 2z = 6$ and the coordinate planes in the first octant.

Sol. $\nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Let $g(x,y,z) = 2x + y + 2z - 6 = 0$, then $\nabla g = \langle 2, 1, 2 \rangle$ and $\hat{n} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

$$\left(\nabla \times \vec{F}\right) \cdot \hat{n} = \frac{2}{3} + \frac{1}{3} + \frac{2}{3} = \frac{5}{3}$$

$$\oint\limits_C F \cdot dr = \iint\limits_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\ dS = \iint\limits_R \frac{5}{3}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA, \text{ where } z = 3 - x - \frac{1}{2}y$$

$$= \iint\limits_R \frac{5}{3}\frac{3}{2} dA = \frac{5}{2} \text{ (area of triangle bounded by lines } x = 0, y = 0 \text{ and } 2x + y = 6)$$

$$= \frac{5}{2}\left(\frac{6 \times 3}{2}\right) = \frac{45}{2}.$$

Q:7 (15 points) Let $\vec{F}(x,y,z) = x^3\ \mathbf{i} + y^3\ \mathbf{j} + z^3\ \mathbf{k}$ and D is the region bounded by the sphere $x^2 + y^2 + z^2 = 9$. Use divergence theorem to evaluate $\iint\limits_S (F \cdot n)\, ds$.

Sol. $\nabla \cdot \vec{F} = \dfrac{\partial (x^3)}{\partial x} + \dfrac{\partial (y^3)}{\partial y} + \dfrac{\partial (z^3)}{\partial z} = 3\left(x^2 + y^2 + z^2\right).$

$$\iint\limits_S (F \cdot n)\, ds = \iiint\limits_D \nabla \cdot \vec{F} dV = \iiint\limits_D 3\left(x^2 + y^2 + z^2\right) dV$$

$$= 3\int_0^{\pi}\int_0^{2\pi}\int_0^{3} \rho^2\ \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{2916}{5}\pi.$$

Q:8 (a) (5 points) Find Laplace Transform for $f(t) = \cos(kt)$.

(b) (5 points) Find Laplace Transform for $g(t) = \cos^2(t)$.

Sol. (a) $\mathcal{L}\{\cos kt\} = \int\limits_0^\infty \cos kt e^{-st} dt = \frac{\sin kt}{k} e^{-st}\Big|_0^\infty - \int\limits_0^\infty \frac{\sin kt}{k}(-s) e^{-st} dt$

$$= \frac{s}{k}\left[\frac{-\cos kt}{k} e^{-st}\Big|_0^\infty - \int\limits_0^\infty \frac{-\cos kt}{k}(-s) e^{-st} dt\right]$$

$$= \frac{s}{k}\left(0 + \frac{1}{k}\right) - \frac{s^2}{k^2}\mathcal{L}\{\cos kt\}$$

$$\Rightarrow \left(\frac{s^2+k^2}{k^2}\right)\mathcal{L}\{\cos kt\} = \frac{s}{k^2} \quad \Rightarrow \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}.$$

(b) $\mathcal{L}\{\cos^2 t\} = \mathcal{L}\left(\frac{1}{2} + \frac{\cos 2t}{2}\right) = \mathcal{L}\left\{\frac{1}{2}\right\} + \frac{1}{2}\mathcal{L}\{\cos 2t\} = \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+k^2}.$