Solution Math 202 Quiz 7 (A)

Q.1: Solve the linear system $X' = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Sol: Eigenvalues are given by $\begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = 0, \Rightarrow \lambda = 1, -1.$

For $\lambda = 1$, solve $(A - \lambda I) K = O \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

For $\lambda = -1$, solve $(A - \lambda I) K = O \Rightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ -1 & -1 & 0 \end{array}\right] \Rightarrow K_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The two $X_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t, \; X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

The fundamental matrix is $\Phi(t) = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$ and $\Phi^{-1} = -\frac{1}{2}\begin{bmatrix} e^{-t} & e^{-t} \\ -e^t & -3e^t \end{bmatrix}$.

$$U(t) = \int \Phi^{-1} F \, dt = \int -\frac{1}{2}\begin{bmatrix} e^{-t} & e^{-t} \\ -e^t & -3e^t \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} dt = \int -\frac{1}{2}\begin{bmatrix} -e^{-t} \\ 5e^t \end{bmatrix} dt = -\frac{1}{2}\begin{bmatrix} e^{-t} \\ 5e^t \end{bmatrix}$$

$$X_p = \Phi(t) U(t) = -\frac{1}{2}\begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} e^{-t} \\ 5e^t \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

The general solution is $X(t) = X_c + X_p = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

Q.2: Show that the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ is NILPOTENT. Compute e^{At} and write solution of the linear system $X' = AX$.

Sol: $A^2 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} t + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \frac{t^2}{2}$$

$$= \begin{bmatrix} 1 + t + \frac{1}{2}t^2 & \frac{1}{2}t^2 & t + \frac{1}{2}t^2 \\ -t & -t + 1 & -t \\ -\frac{1}{2}t^2 & t - \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 \end{bmatrix}$$

$$X(t) = e^{At}C = \begin{bmatrix} 1 + t + \frac{1}{2}t^2 & \frac{1}{2}t^2 & t + \frac{1}{2}t^2 \\ -t & -t + 1 & -t \\ -\frac{1}{2}t^2 & t - \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 1 + t + \frac{1}{2}t^2 \\ -t \\ -\frac{1}{2}t^2 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{2}t^2 \\ -t + 1 \\ t - \frac{1}{2}t^2 \end{bmatrix} + c_3 \begin{bmatrix} t + \frac{1}{2}t^2 \\ -t \\ 1 - \frac{1}{2}t^2 \end{bmatrix}$$