Solution Math 202 Quiz 6 (B)

Q.1: Solve the linear system $X' = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} -1-\lambda & 4 & 2 \\ 4 & -1-\lambda & -2 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 27\lambda + 4\lambda^2 - \lambda^3 - 90 = 0$

$27\lambda + 4\lambda^2 - \lambda^3 - 90 = 0$, Solution is: $\lambda = 3, -5, 6$.

For $\lambda = 3$, solve $(A - \lambda I) K = O \Rightarrow \left[\begin{array}{ccc|c} -4 & 4 & 2 & 0 \\ 4 & -4 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = -5$, solve $(A - \lambda I) K = O \Rightarrow \left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & -2 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right] \Rightarrow K_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 6$, solve $(A - \lambda I) K = O \Rightarrow \left[\begin{array}{ccc|c} -7 & 4 & 2 & 0 \\ 4 & -7 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R2} \left[\begin{array}{ccc|c} -7 & 4 & 2 & 0 \\ -3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} -7 & 4 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow^{7R_2+R_1} \left[\begin{array}{ccc|c} 0 & 11 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad K_3 = \begin{bmatrix} 2 \\ -2 \\ 11 \end{bmatrix}$

The three solutions are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t},\ X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t},\ X_3 = \begin{bmatrix} 2 \\ -2 \\ 11 \end{bmatrix} e^{6t}$.

Q.2: Solve the linear system $X' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 3\lambda^2 - \lambda - \lambda^3 - 1 = 0$

$3\lambda^2 - \lambda - \lambda^3 - 1 = 0$, Solution is: $\lambda = 1, 1 - \sqrt{2}, 1 + \sqrt{2}$

For $\lambda = 1$, solve $(A - \lambda I) K = O \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 1 + \sqrt{2}$, solve $(A - \lambda I) K = O \Rightarrow \left[\begin{array}{ccc|c} -\sqrt{2} & 0 & 0 & 0 \\ 2 & 1-\sqrt{2} & 1 & 0 \\ 0 & 1 & -1-\sqrt{2} & 0 \end{array}\right]$

$$\Rightarrow K_2 = \begin{bmatrix} 0 \\ 1+\sqrt{2} \\ 1 \end{bmatrix}$$

For $\lambda = 1-\sqrt{2}$, solve $(A-\lambda I)\,K = O \ \Rightarrow \left[\begin{array}{ccc|c} \sqrt{2} & 0 & 0 & 0 \\ 2 & 1+\sqrt{2} & 1 & 0 \\ 0 & 1 & -1+\sqrt{2} & 0 \end{array}\right]$

$$\Rightarrow K_2 = \begin{bmatrix} 0 \\ 1-\sqrt{2} \\ 1 \end{bmatrix}$$

The three solutions are $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t,\ X_2 = \begin{bmatrix} 0 \\ 1+\sqrt{2} \\ 1 \end{bmatrix} e^{\left(1+\sqrt{2}\right)t},\ X_3 = \begin{bmatrix} 0 \\ 1-\sqrt{2} \\ 1 \end{bmatrix} e^{\left(1-\sqrt{2}\right)t}.$

Q.3: Solve the linear system $X' = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} X.$

Sol: Eigenvalues are given by $\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 3\lambda^2 - 4\lambda - \lambda^3 + 2 = 0$

$3\lambda^2 - 4\lambda - \lambda^3 + 2 = 0$, Solution is: $\lambda = 1, 1-i, 1+i$

For $\lambda = 1$, solve $(A-\lambda I)\,K = O \ \Rightarrow \left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

For $\lambda = 1+i$, solve $(A-\lambda I)\,K = O \ \Rightarrow \left[\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array}\right] \overset{\substack{-iR_2+R_1 \\ -R_2+R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 0 & -2 & 2 & 0 \\ -1 & -i & 0 & 0 \\ 0 & i & -i & 0 \end{array}\right]$

$$K_2 = \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix}$$

$B_1 = \operatorname{Re}(K_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $B_2 = \operatorname{Im}(K_2) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

The three solutions are $X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{6t}, \quad X_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \sin t \right\} e^t = \begin{bmatrix} \sin t \\ \cos t \\ \cot s \end{bmatrix} e^t,$

$$X_3 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \sin t \right\} e^t = \begin{bmatrix} -\cos t \\ \sin t \\ \sin t \end{bmatrix} e^t.$$