Solution Math 202 Quiz 6 (A)

Q.1: Solve the linear system $X' = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = -(2-\lambda) + (2-\lambda)\left((2-\lambda)^2 + 1\right) = 0$

$(2-\lambda)(2-\lambda)^2 = 0 \quad \Rightarrow \lambda = 2, 2, 2.$

For $\lambda = 2$, solve $(A - \lambda I)K = O \ \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

To find vector P, solve $(A - \lambda I)P = K \ \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow P = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

To find vector Q, solve $(A - \lambda I)Q = P \ \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow Q = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

The three solutions are $X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t}$, $X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$,

$$X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} = \begin{bmatrix} t+1 \\ -\frac{t^2}{2} - t + 1 \\ \frac{t^2}{2} + t \end{bmatrix} e^{2t}.$$

Q.2: Solve the linear system $X' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1 - 2i, 1 + 2i$

For $\lambda = 1 + 2i$, solve $(A - \lambda I)K = O \ \Rightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$

The eigenvector is $K = \begin{bmatrix} -i \\ 1 \end{bmatrix}$. Thus $B_1 = \text{Re}(K) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $B_2 = \text{Im}(K) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The two solutions are $X_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin 2t \right\} e^t = \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} e^t$,

$$X_3 = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t \right\} e^t = \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix} e^t.$$