Solution Math 202 Quiz 5 (B)

Q.1: Show that $x = 0$ is a regular singular point of the differential equation $4xy'' + \frac{1}{2}y' + y = 0$. Find the two indicial roots of the singularity. If they donot differ by an integer, then find two linearly independent series solutions about $x = 0$ and form the general solution.

Sol: Given equation can be written as $y'' + \frac{1}{8x}y' + \frac{1}{4x}y = 0$.

So $P(x) = \frac{1}{8x}$, $Q(x) = \frac{1}{4x}$ $\Rightarrow x = 0$ is a singular point.

Since both $xP(x) = \frac{1}{8}$ and $x^2Q(x) = \frac{1}{4}x$ are analytic at $x = 0$.

Therefore $x = 0$ is a regular singular point.

$xP(x) = \frac{1}{8}$ $\Rightarrow a_0 = \frac{1}{8}$ and $x^2Q(x) = \frac{1}{4}x$ $\Rightarrow b_0 = 0$.

The indicial equation is $r(r-1) + a_0 r + b_0 = 0$

$\Rightarrow r^2 - r + \frac{1}{8}r = 0 \Rightarrow r^2 - \frac{7}{8}r = 0 \Rightarrow r = 0, \frac{7}{8}$ are the indicial roots.

Now let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, then $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$

Substituting in given equation, we get

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} \frac{1}{2}(n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)\left[(4n+4r-4) + \frac{1}{2}\right] c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$x^r \left\{ \sum_{n=0}^{\infty} (n+r)\left(4n+4r-4+\frac{1}{2}\right) c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \right\} = 0$$

$$x^r \left\{ r(4r - \frac{7}{2})c_0 x^{-1} + \sum_{n=1}^{\infty} (n+r)\left(4n+4r-4+\frac{1}{2}\right) c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \right\} = 0$$

Substitute $k = n-1$ in the first series and $k = n$ in the second series,

$$x^r \left\{ r(4r - \frac{7}{2})c_0 x^{-1} + \sum_{k=0}^{\infty} (k+1+r)\left(4k+4+4r-4+\frac{1}{2}\right) c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k \right\} = 0$$

$$x^r \left\{ r(4r - \frac{7}{2})c_0 x^{-1} + \sum_{k=0}^{\infty} \left[(k+1+r)\left(4k+4r+\frac{1}{2}\right) c_{k+1} + c_k\right] x^k \right\} = 0$$

Comparing the coefficients of powers of x we get

$$r(4r - \frac{7}{2}) = 0 \;\Rightarrow r = 0, \frac{7}{8} \;\text{ and } c_{k+1} = -\frac{c_k}{(k+1+r)\left(4k+4r+\frac{1}{2}\right)}$$

For $r = 0$, $c_{k+1} = -\dfrac{2c_k}{(k+1)(8k+1)}$

$$k = 0 \;\Rightarrow c_1 = \frac{-2}{1}c_0 = -2c_0$$

$$k = 1, \;\Rightarrow c_2 = -\frac{2}{2\,(9)}c_1 = \frac{2}{9}c_0$$

$$k = 2, \;\Rightarrow c_3 = -\frac{2}{3\,(17)}c_2 = -\frac{4}{3.9.17}c_0$$

For $r = \frac{7}{8}$, $c_{k+1} = -\dfrac{c_k}{\left(k+1+\frac{7}{8}\right)\left(4k+\frac{7}{2}+\frac{1}{2}\right)} = -\dfrac{2c_k}{(8k+15)(k+1)}$

$$k = 0 \;\Rightarrow c_1 = -\frac{2}{15}c_0$$

$$k = 1, \;\Rightarrow c_2 = -\frac{2}{23\,(2)}c_1 = \frac{2}{15.23}c_0$$

$$k = 2, \;\Rightarrow c_3 = -\frac{2}{31\,(3)}c_2 = -\frac{4}{3.15.23.31}c_0$$

$$y = C\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{3.9.17}x^3 + ...\right) + Dx^{\frac{7}{8}}\left(1 - \frac{2}{15}x + \frac{2}{15.23}x^2 - \frac{4}{3.15.23.31}x^3 + ...\right)$$

Q.2: Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$.

Sol: The eigenvalues are given by $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ 5 & 2-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)\left((2-\lambda)^2 - 4\right) + 5(2-\lambda) = (2-\lambda) \quad (\lambda^2 - 4\lambda + 5) = 0$$

$$\Rightarrow \lambda = 2,\ 2+i,\ 2-i.$$

For $\lambda = 2$, solve $(A - 2I)K = 0$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] 0 \quad \Rightarrow y = 0 \text{ and } x = -\frac{4}{5}z$$

So $K_1 = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 2$

For $\lambda = 2+i$, solve the system $(A - (2+i)I)K = 0$

$$\left[\begin{array}{ccc|c} -i & -1 & 0 & 0 \\ 5 & -i & 4 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \quad -5iR_1 + R_2 \Rightarrow \left[\begin{array}{ccc|c} -i & -1 & 0 & 0 \\ 0 & 4i & 4 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \Rightarrow x = iy \text{ and } z = -iy$$

So $K_2 = \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}$. Similarly for $\lambda = 2-i$, $K_3 = \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}$.