Solution Math 202 Quiz 5

Q.1: Solve Cauchy-Euler equation

$$x^3y''' + 4x^2y'' + 5xy' + 3y = 0.$$

Sol: Let $y = x^m$, then $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$, and $y''' = m(m-1)(m-2)x^{m-3}$.

$$m(m-1)(m-2) + 4m(m-1) + 5m + 3 = 0 \Rightarrow m^3 + m^2 + 3m + 3 = 0 \Rightarrow m = -1, \pm\sqrt{3}i$$

$$y = C_1x^{-1} + C_2\cos\left(\sqrt{3}\ln x\right) + C_3\sin\left(\sqrt{3}\ln x\right).$$

Q.2: Find two power series solutions of the DE $y'' + 2xy' + y = 0$ about the ordinary point $x = 0$.

Sol: Let $y = \sum_{n=0}^{\infty} c_nx^n$, then $y' = \sum_{n=1}^{\infty} nc_nx^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + 2\sum_{n=1}^{\infty} nc_nx^n + \sum_{n=0}^{\infty} c_nx^n = 0$$

$$2c_2 + \sum_{n=3}^{\infty} n(n-1)c_nx^{n-2} + 2\sum_{n=1}^{\infty} nc_nx^n + c_0 + \sum_{n=1}^{\infty} c_nx^n = 0$$

$$2c_2 + c_0 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k + 2\sum_{k=1}^{\infty} kc_kx^k + \sum_{k=1}^{\infty} c_kx^k = 0$$

$$2c_2 + c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2}x^k + (2k+1)c_k\right]x^k = 0$$

$$2c_2 + c_0 = 0 \Rightarrow c_2 = -\frac{c_0}{2} \text{ and } c_{k+2} = \frac{-(2k+1)c_k}{(k+2)(k+1)},\ k = 1, 2, 3, ...$$

$$k = 1,\ c_3 = \frac{-3}{3\cdot 2}c_1 = -\frac{1}{2}c_1$$

$$k = 2,\ c_4 = \frac{-5}{4\cdot 3}c_2 = \frac{5}{24}c_0$$

$$k = 3,\ c_5 = \frac{-7}{5\cdot 4}c_3 = \frac{7}{40}c_1$$

$$y = c_0 + c_1x + c_2x^2 + +c_3x^3 + c_4x^4 + c_5x^5 + ...$$

$$= c_0\left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + ...\right) + c_1\left(x - \frac{1}{2}x^3 + \frac{7}{40}x^5 + ...\right)$$