Solution Math 202 Quiz 5 (A)

Q.1: Show that $x = 0$ is a regular singular point of the differential equation $2xy'' - (3+2x)\,y' + y = 0$. Find the two indicial roots of the singularity. If they donot differ by an integer, then find two linearly independent series solutions about $x = 0$ and form the general solution.

Sol: Given equation can be written as $y'' - \dfrac{(3+2x)}{2x}y' + \dfrac{1}{2x}y = 0.$

So $P(x) = -\dfrac{3}{2x} - 1,\ Q(x) = \dfrac{1}{2x} \Rightarrow x = 0$ is a singular point.

Since both $xP(x) = -\dfrac{3}{2} - x$ and $x^2Q(x) = \dfrac{1}{2}x$ are analytic at $x = 0$.

Therefore $x = 0$ is a regular singular point.

$xP(x) = -\dfrac{3}{2} - x \ \Rightarrow a_0 = -\dfrac{3}{2}$ and $x^2Q(x) = \dfrac{1}{2}x \ \Rightarrow b_0 = 0.$

The indicial equation is $r(r-1) + a_0 r + b_0 = 0$

$\Rightarrow r^2 - r - \dfrac{3}{2}r = 0 \ \Rightarrow r^2 - \dfrac{5}{2}r = 0 \ \Rightarrow r = 0, \dfrac{5}{2}$ are the indicial roots.

Now let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, then $y' = \sum_{n=0}^{\infty} (n+r)\,c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\,c_n x^{n+r-2}$

Substituting in given equation, we get

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)\,c_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r)\,c_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)\,c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) - 3(n+r)]\,c_n x^{n+r-1} - \sum_{n=0}^{\infty} [2(n+r) - 1]\,c_n x^{n+r} = 0$$

$$x^r \left\{ \sum_{n=0}^{\infty} [2(n+r)(n+r-1) - 3(n+r)]\,c_n x^{n-1} - \sum_{n=0}^{\infty} [2(n+r) - 1]\,c_n x^{n} \right\} = 0$$

$$x^r \left\{ r(2r-5)c_0 x^{-1} + \sum_{n=1}^{\infty} [(n+r)(2n+2r-2-3)]\,c_n x^{n-1} - \sum_{n=0}^{\infty} [2(n+r) - 1]\,c_n x^{n} \right\} = 0$$

Substitute $k = n-1$ in the first series and $k = n$ in the second series,

$$x^r \left\{ r(2r-5)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+1+r)(2k+2+2r-5)]\,c_{k+1} x^{k} - \sum_{k=0}^{\infty} [2(k+r) - 1]\,c_k x^{k} \right\} = 0$$

$$x^r \left\{ r(2r-5)c_0 x^{-1} + \sum_{k=0}^{\infty} \left([(k+1+r)(2k+2+2r-5)]\,c_{k+1} - [2k+2r-1]\,c_k \right) x^{k} \right\} = 0$$

Comparing the coefficients of powers of x we get

$$r(2r-5)=0 \;\Rightarrow r=0,\frac{5}{2} \;\text{ and } c_{k+1}=\frac{(2k+2r-1)\,c_k}{(k+1+r)\,(2k+2r-3)}$$

For $r=0,\; c_{k+1}=\dfrac{(2k-1)\,c_k}{(k+1)\,(2k-3)}$

$$k=0 \;\Rightarrow c_1=\frac{-1}{-3}c_0=\frac{1}{3}c_0$$

$$k=1,\;\Rightarrow c_2=\frac{1}{2\,(-1)}c_1=-\frac{1}{6}c_0$$

$$k=2,\;\Rightarrow c_3=\frac{3}{3}c_2=-\frac{1}{6}c_0$$

For $r=\frac{5}{2},\; c_{k+1}=\dfrac{(2k+4)\,c_k}{(2k+7)\,(k+1)}$

$$k=0 \;\Rightarrow c_1=\frac{4}{7}c_0$$

$$k=1,\;\Rightarrow c_2=\frac{6}{9\,(2)}c_1=\frac{1}{3}\frac{4}{7}c_0=\frac{1.4}{3.7}$$

$$k=2,\;\Rightarrow c_3=\frac{8}{11\,(3)}c_2=\frac{8}{11\,(3)}\frac{1}{3}\frac{4}{7}c_0=\frac{4.8}{7.9.11}c_0$$

$$y=C\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3+...\right)+Dx^{\frac{5}{2}}\left(1+\frac{4}{7}x+\frac{4}{3.7}x^2+\frac{4.8}{7.9.11}x^3+...\right)$$

Q.2: Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

Sol: The eigenvalues are given by $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 5 & 1 \\ -5 & -6-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)\left((-6-\lambda)(2-\lambda)\right) + 25(2-\lambda) = (2-\lambda) \qquad (\lambda^2 + 4\lambda + 13) = 0$$

$$\Rightarrow \lambda = 2, \ -2+3i, \ -2-3i.$$

For $\lambda = 2$, solve $(A - 2I)K = 0$

$$\left[\begin{array}{ccc|c} 0 & 5 & 1 & 0 \\ -5 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad -4R_1 + R2 \ \Rightarrow \left[\begin{array}{ccc|c} 0 & 5 & 1 & 0 \\ -5 & -28 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow z = -5y \text{ and } x = -\frac{28}{5}y$$

So $K_1 = \begin{bmatrix} -28 \\ 5 \\ -25 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 2$

For $\lambda = -2+3i$, solve the system $(A + (2-3i)I)K = 0$

$$\left[\begin{array}{ccc|c} 4-3i & 5 & 1 & 0 \\ -5 & -4-3i & 4 & 0 \\ 0 & 0 & 4-3i & 0 \end{array}\right] \quad \Rightarrow z = 0 \text{ and } y = -\frac{4-3i}{5}x$$

So $K_2 = \begin{bmatrix} -5 \\ 4-3i \\ 0 \end{bmatrix}$. Similarly for $\lambda = -2-3i$, $K_3 = \begin{bmatrix} -5 \\ 4+3i \\ 0 \end{bmatrix}$.