Solution Math 202 Quiz 4 (B)

Q.1: Solve the initial value problem $4y'' - y = xe^{\frac{x}{2}} \quad y(0) = 1, \quad y'(0) = 0.$

Sol: The auxiliary equation of the corresponding homogeneous equation is

$4m^2 - 1 = 0 \ \Rightarrow (2m+1)(2m-1) = 0 \ \Rightarrow m = \frac{1}{2}, -\frac{1}{2}$

The complementary function is $y_c = C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{x}{2}}$. Let $y_1(x) = e^{\frac{x}{2}}$ and $y_2(x) = e^{-\frac{x}{2}}$

$$W = \begin{vmatrix} e^{\frac{x}{2}} & e^{-\frac{x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & -\frac{1}{2}e^{-\frac{x}{2}} \end{vmatrix} = -1,\ W_1 = \begin{vmatrix} 0 & e^{-\frac{x}{2}} \\ \frac{x}{4}e^{\frac{x}{2}} & -\frac{1}{2}e^{-\frac{x}{2}} \end{vmatrix} = -\frac{1}{4}x,\ W_2 = \begin{vmatrix} e^{\frac{x}{2}} & 0 \\ \frac{1}{2}e^{\frac{x}{2}} & \frac{x}{4}e^{\frac{x}{2}} \end{vmatrix} = \frac{1}{4}xe^{x}$$

$u_1(x) = \int \frac{W_1}{W}dx = \int \frac{x}{4}dx = \frac{x^2}{8}$

and $u_2(x) = \int \frac{W_2}{W}dx = \int -\frac{1}{4}xe^{x}dx = \frac{1}{4}e^{x} - \frac{1}{4}xe^{x}$

So $y_p = y_1u_1 + y_2u_2 = \frac{x^2}{8}e^{\frac{x}{2}} + \frac{1}{4}e^{\frac{x}{2}} - \frac{1}{4}xe^{\frac{x}{2}}$

The general solution is $y = C_1e^{\frac{x}{2}} + C_2e^{-\frac{x}{2}} + \frac{x^2}{8}e^{\frac{x}{2}} + \frac{1}{4}e^{\frac{x}{2}} - \frac{1}{4}xe^{\frac{x}{2}}$

$y(0) = 1 \ \Rightarrow 1 = C_1 + C_2 + \frac{1}{4} \ \Rightarrow C_1 + C_2 = \frac{3}{4}$

$y' = \frac{1}{2}C_1e^{\frac{x}{2}} - \frac{1}{2}C_2e^{-\frac{x}{2}} + \frac{x}{4}e^{\frac{x}{2}} + \frac{x^2}{16}e^{\frac{x}{2}} + \frac{1}{8}e^{\frac{x}{2}} - \frac{1}{4}e^{\frac{x}{2}} - \frac{1}{8}xe^{\frac{x}{2}}$

and $y'(0) = 0 \ \Rightarrow 0 = \frac{1}{2}C_1 - \frac{1}{2}C_2 + \frac{1}{8} - \frac{1}{4} \ \Rightarrow C_1 - C_2 = \frac{1}{4}$

Solving $C_1 + C_2 = \frac{3}{4}$ and $C_1 - C_2 = \frac{1}{4}$

we get $2C_1 = 1 \ \Rightarrow C_1 = \frac{1}{2}$ and $C_2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Thus $y = \frac{1}{2}e^{\frac{x}{2}} - \frac{1}{4}e^{-\frac{x}{2}} + \frac{x^2}{8}e^{\frac{x}{2}} + \frac{1}{4}e^{\frac{x}{2}} - \frac{1}{4}xe^{\frac{x}{2}}$

$\quad = \frac{3}{4}e^{\frac{x}{2}} - \frac{1}{4}e^{-\frac{x}{2}} + \frac{x^2}{8}e^{\frac{x}{2}} - \frac{1}{4}xe^{\frac{x}{2}}$

Q.2: Solve the differential equation $x^2y'' - 3xy' + 3y = 2x^4e^x$.

Sol: This is a Cauchy-Euler equation and auxiliary equation for the corresponding homogeneous equation is $m(m-1) - 3m + 3 = 0 \;\Rightarrow m^2 - 4m + 3 = 0 \;\Rightarrow m = 1, 3$.
The complementary function is $y_c = C_1x + C_2x^3$. Let $y_1 = x$ and $y_2 = x^3$
Standard form of the given equation is $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \Rightarrow W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \text{ and } W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x.$$

$$u_1 = \int \frac{W_1}{W}dx = \int \frac{-2x^5e^x}{2x^3}dx = \int -x^2e^x dx = 2xe^x - 2e^x - x^2e^x$$

and $u_2 = \int \frac{W_2}{W}dx = \int \frac{2x^3e^x}{2x^3}dx = e^x$

$$y_p = 2x^2e^x - 2xe^x - x^3e^x + x^3e^x = 2x^2e^x - 2xe^x$$

$$y = C_1x + C_2x^3 + 2x^2e^x - 2xe^x.$$

Q.3: Add the power series $x^2\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + 4\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} - 2x\sum_{n=1}^{\infty} nc_nx^{n-1}$

Sol:

$$x^2\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + 4\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} - 2x\sum_{n=1}^{\infty} nc_nx^{n-1}$$

$$= \sum_{n=2}^{\infty} n(n-1)c_nx^n + \sum_{n=2}^{\infty} 4n(n-1)c_nx^{n-2} - \sum_{n=1}^{\infty} 2nc_nx^n$$

$$= \sum_{n=2}^{\infty} n(n-1)c_nx^n + 8c_2 + 24c_3x + \sum_{n=4}^{\infty} 4n(n-1)c_nx^{n-2} - 2c_1x - \sum_{n=2}^{\infty} 2nc_nx^n$$

$$= 8c_2 + 24c_3x - 2c_1x + \sum_{k=2}^{\infty} k(k-1)c_kx^k + \sum_{k=2}^{\infty} 4(k+2)(k+1)c_{k+2}x^k - \sum_{k=2}^{\infty} 2kc_kx^k$$

$$= 8c_2 + 24c_3x - 2c_1x + \sum_{k=2}^{\infty} \left[k(k-3)c_k + 4(k+2)(k+1)c_{k+2}\right]x^k.$$

Q.2: (for Sec 15) Solve the differential equation $x^2y'' - 3xy' + 3y = 2x^4$.

Sol: This is a Cauchy-Euler equation and auxiliary equation for the corresponding homogeneous equation is $m(m-1) - 3m + 3 = 0 \;\Rightarrow m^2 - 4m + 3 = 0 \;\Rightarrow m = 1, 3$.
The complementary function is $y_c = C_1x + C_2x^3$. Let $y_1 = x$ and $y_2 = x^3$
Standard form of the given equation is $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \Rightarrow W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2 & 3x^2 \end{vmatrix} = -2x^5, \text{ and } W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2 \end{vmatrix} = 2x^3$$

$$u_1 = \int \frac{W_1}{W}dx = \int \frac{-2x^5}{2x^3}dx = \int -x^2dx = -\frac{1}{3}x^3$$

and $u_2 = \int \frac{W_2}{W}dx = \int \frac{2x^3}{2x^3}dx = x$

$y_p = -\frac{1}{3}x^4 + x^4 = \frac{2}{3}x^4$ and $y = C_1x + C_2x^3 + \frac{2}{3}x^4.$