Solution Math 202 Quiz 4 (A)

Q.1: Solve the initial value problem $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$, $y(0) = 1$, $y'(0) = 0$.

Sol: The auxiliary equation of the corresponding homogeneous equation is

$m^2 + 2m - 8 = 0 \Rightarrow (m+4)(m-2) = 0 \Rightarrow m = 2, -4.$

The complementary function is $y_c = C_1 e^{2x} + C_2 e^{-4x}$. Let $y_1(x) = e^{2x}$ and $y_2(x) = e^{-4x}$.

Then $W = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix} = -6e^{-2x}$, $W_1 = \begin{vmatrix} 0 & e^{-4x} \\ 2e^{-2x} - e^{-x} & -4e^{-4x} \end{vmatrix} = e^{-5x} - 2e^{-6x}$,

$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{-2x} - e^{-x} \end{vmatrix} = 2 - e^{x}$

$$u_1(x) = \int \frac{W_1}{W} dx = \int \frac{e^{-5x} - 2e^{-6x}}{-6e^{-2x}} dx = \frac{e^{-3x}}{18} - \frac{e^{-4x}}{12}$$

and $u_2(x) = \int \frac{W_2}{W} dx = \int \frac{2 - e^{x}}{-6e^{-2x}} dx = \frac{e^{3x}}{18} - \frac{e^{2x}}{6}$

So $y_p = y_1 u_1 + y_2 u_2 = \frac{e^{-x}}{18} - \frac{e^{-2x}}{12} + \frac{e^{-x}}{18} - \frac{e^{-2x}}{6} = \frac{1}{9}e^{-x} - \frac{1}{4}e^{-2x}$

The general solution is $y = C_1 e^{2x} + C_2 e^{-4x} + \frac{1}{9}e^{-x} - \frac{1}{4}e^{-2x}$

$y(0) = 1 \Rightarrow 1 = C_1 + C_2 + \frac{1}{9} - \frac{1}{4} \Rightarrow C_1 + C_2 = \frac{41}{36} \Rightarrow 4C_1 + 4C_2 = \frac{164}{36}$

$y' = 2C_1 e^{2x} - 4C_2 e^{-4x} - \frac{1}{9}e^{-x} + \frac{1}{2}e^{-2x}$

and $y'(0) = 0 \Rightarrow 0 = 2C_1 - 4C_2 - \frac{1}{9} + \frac{1}{2} \Rightarrow 2C_1 - 4C_2 = -\frac{7}{18}$

Solving $4C_1 + 4C_2 = \frac{164}{36}$ and $2C_1 - 4C_2 = -\frac{7}{18}$

we get $6C_1 = \frac{164 - 14}{36} = \frac{25}{6} \Rightarrow C_1 = \frac{25}{36}$ and $C_2 = \frac{41}{36} - \frac{25}{36} = \frac{4}{9}$

Thus $y = \frac{25}{36}e^{2x} + \frac{4}{9}e^{-4x} + \frac{1}{9}e^{-x} - \frac{1}{4}e^{-2x}$.

Q.2: Solve the differential equation $x^2y'' - xy' + y = 2x$.

Sol: This is a Cauchy-Euler equation and auxiliary equation for the corresponding homogeneous equation is $m(m-1) - m + 1 = 0 \Rightarrow m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$.

The complementary function is $y_c = C_1x + C_2x\ln x$. Let $y_1 = x$ and $y_2 = x\ln x$

Standard form of the given equation is $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x}$

$$W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x, \Rightarrow W_1 = \begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & \ln x + 1 \end{vmatrix} = -2\ln x, \text{ and } W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix} = 2$$

$$u_1 = \int \frac{W_1}{W}dx = \int \frac{-2\ln x}{x}dx = -(\ln x)^2 \text{ and } u_2 = \int \frac{W_2}{W}dx = \int \frac{2}{x}dx = 2\ln x$$

$$y_p = -x(\ln x)^2 + 2x(\ln x)^2 = x(\ln x)^2$$

$$y = C_1x + C_2x\ln x + x(\ln x)^2.$$

Q.3: Add the power series $x\sum_{n=2}^{\infty} n(n-1)c_nx^{n-1} + 2\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + 3x^2\sum_{n=1}^{\infty} nc_nx^{n-2}$

Sol:

$$x\sum_{n=2}^{\infty} n(n-1)c_nx^{n-1} + 2\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + 3x^2\sum_{n=1}^{\infty} nc_nx^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1)c_nx^n + \sum_{n=2}^{\infty} 2n(n-1)c_nx^{n-2} + \sum_{n=1}^{\infty} 3nc_nx^n$$

$$= \sum_{n=2}^{\infty} n(n-1)c_nx^n + 4c_2 + 12c_3x + \sum_{n=4}^{\infty} 2n(n-1)c_nx^{n-2} + 3c_1x + \sum_{n=2}^{\infty} 3nc_nx^n$$

$$= 4c_2 + 12c_3x + 3c_1x + \sum_{k=2}^{\infty} k(k-1)c_kx^k + \sum_{k=2}^{\infty} 2(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} 3kc_kx^k$$

$$= 4c_2 + 12c_3x + 3c_1x + \sum_{k=2}^{\infty} [k(k+2)c_k + 2(k+2)(k+1)c_{k+2}]x^k.$$