Solution Math 202-091 Quiz 3 (B)

Q.1: If $y_1(x) = x + 2$ is a solution of $(5 - 4x - x^2)\, y'' + 2(2 + x)\, y' - 4y = 0$, then find a second solution $y_2(x)$.

Sol: Given equation can be written as $y'' - \dfrac{4 + 2x}{x^2 + 4x - 5} y' + \dfrac{4}{x^2 + 4x - 5} y = 0.$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx = (x + 2) \int \frac{e^{\int \frac{2x+4}{x^2+4x-5} dx}}{(x + 2)^2} dx$$

$$= (x + 2) \int \frac{e^{\ln\left(x^2+4x-5\right)}}{(x + 2)^2} dx = (x + 2) \int \frac{x^2 + 4x - 5}{x^2 + 4x + 4} dx$$

$$= (x + 2) \int \left(1 - \frac{9}{x^2 + 4x + 4}\right) dx = (x + 2) \int \left(1 - \frac{9}{(x + 2)^2}\right) dx$$

$$= (x + 2) \left(x + \frac{9}{x + 2}\right) = x^2 + 2x + 9.$$

Q.2: Solve the initial value problem $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 1.$

Sol: Auxiliary equation is $m^3 + 2m^2 - 5m - 6 = (m - 2)(m + 3)(m + 1) = 0 \ \Rightarrow m = -1, 2, -3.$

The solution is $y = Ae^{-x} + Be^{2x} + Ce^{-3x}$,

Then $y' = -Ae^{-x} + 2Be^{2x} - 3Ce^{-3x}$

and $y'' = Ae^{-x} + 4Be^{2x} + 9Ce^{-3x}$

$y(0) = 0 \ \Rightarrow A + B + C = 0$

$y'(0) = 1 \ \Rightarrow -A + 2B - 3C = 0$

$y''(0) = 1 \ \Rightarrow A + 4B + 9C = 1$

$3B - 2C = 0$ and $6B + 6C = 1 \ \Rightarrow B = \dfrac{1}{15},\ C = \dfrac{3}{2}B = \dfrac{1}{10},$

$A = -B - C = -\dfrac{1}{15} - \dfrac{1}{10} = -\dfrac{1}{6}$

$y = -\dfrac{1}{6}e^{-x} + \dfrac{1}{15}e^{2x} + \dfrac{1}{10}e^{-3x}.$