Solution Math 202 Quiz 3 (B)

Q.1: If $y_1 = x^2$ is a solution of the differential equation $x^2y'' + 2xy' - 6y = 0$, find a second solution and write the general solution.

Sol: First we write $x^2y'' + 2xy' - 6y = 0$ as $y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$. Then $P(x) = \frac{2}{x}$

$$y_2(x) = y_1(x) \int \frac{e^{\int -P(x)dx}}{(y_1(x))^2} dx = x^2 \int \frac{e^{\int -\frac{2}{x}dx}}{x^4} dx = x^2 \int \frac{x^{-2}}{x^4} dx = x^2 \int \frac{1}{x^6} dx = x^2 \left(\frac{1}{-5x^5} \right) = \frac{1}{-5x^3}$$

So the general solution is $y(x) = C_1x^2 + C_2x^{-3}$

OR: Let $y(x) = u(x)\, y_1(x) = x^2u$, then $y' = 2xu + x^2u'$ and $y'' = 2u + 4xu' + u''x^2$

$$2x^2u + 4x^3u' + x^4u'' + 4x^2u + 2x^3u' - 6x^2u = 0$$

$$6x^3u' + x^4u'' = 0 \ \Rightarrow xu'' + 6u' = 0.$$

Let $w = u'$ then $w' = u''$ and $xw' = -6w \ \Rightarrow \frac{dw}{w} = -\frac{6}{x}dx \ \Rightarrow \ln(w) = -6\ln(x) \ \Rightarrow w = \frac{1}{x^6}$

$u' = \frac{1}{x^6} \ \Rightarrow u(x) = \frac{1}{-5x^5}$ and $y_2(x) = x^2\left(\frac{1}{-5x^5}\right) = \frac{1}{-5x^3}$.

So the general solution is $y(x) = C_1x^2 + C_2x^{-3}$

Q.2: Solve the initial value problem $y'' - 5y' + 6y = 0,\ y(0) = 0,\ y'(0) = 1$.

Sol: Auxiliary equation is $m^2 - 5m + 6 = 0 \ \Rightarrow m = 2, 3$

$$y(x) = C_1e^{2x} + C_2e^{3x}$$

$$y'(x) = 2C_1e^{2x} + 3C_2e^{3x}$$

$y(0) = 1 \ \Rightarrow 0 = C_1 + C_2$ and $y'(0) = 1 \ \Rightarrow 1 = 2C_1 + 3C_2 = 2C_1 - 3C_1$

$\Rightarrow C_1 = -1$ and $C_2 = 1$

$$y(x) = -e^{2x} + e^{3x}$$

Q.3: Solve $y'' + 4y' + 3y = 3x - 5$ using Undetermined coefficients (Annihilator Approach).

Sol: Auxiliary equation of the corresponding equation is $m^2+4m+3=0 \;\Rightarrow m=-1,-3$

$y_c(x)=C_1e^{-x}+C_2e^{-3x}$

$(D^2+3D+2)\,y=3x-5 \;\Rightarrow D^2(D^2+3D+2)\,y=D^2(3x-5)=0.$

Auxiliary equation of the last fourth order homogeneous equation is

$m^2(m^2+4m+3)=0 \Rightarrow m=0,0,-1,-3 \;\Rightarrow y(x)=C_1+C_2x+C_3e^{-x}+C_4e^{-3x}$

$y_p(x)=C_1+C_2x.$

Q.4: If $m_1=2$ and $m_2=2+i$ are roots of a cubic auxiliary equation of a third order differential equation, find the corresponding homogeneous linear differential equation.

Sol: If $m_2=2+i$ is a complex root then its conjugate $m_3=2-i$ is also a root.

$(m-2)(m-2+i)(m-2-i)=0 \;\Rightarrow (m-2)\left((m-2)^2+1\right)=0$

$\Rightarrow m^3-6m^2+13m-10=0$

The corresponding equation is $y'''-6y''+13y'-10y=0.$