

Solution Math 202 Quiz 3

(A)

**Q.1:** If  $y_1 = x^4$  is a solution of the differential equation  $x^2y'' - 7xy' + 16y = 0$ , find a second solution and write the general solution.

**Sol:** First we write  $x^2y'' - 7xy' + 16y = 0$  as  $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0$ . Then  $P(x) = -\frac{7}{x}$

$$y_2(x) = y_1(x) \int \frac{e^{\int -P(x)dx}}{(y_1(x))^2} dx = x^4 \int \frac{e^{\int \frac{7}{x} dx}}{x^8} dx = x^4 \int \frac{x^7}{x^8} dx = x^4 \ln(x).$$

So the general solution is  $y(x) = C_1x^4 + C_2x^4 \ln(x)$

OR: Let  $y(x) = u(x)y_1(x) = x^4u$ , then  $y' = 4x^3u + x^4u'$  and  $y'' = 12x^2u + 4x^3u' + 4x^3u' + x^4u''$

$$12x^2u + 8x^3u' + x^4u'' - 28x^2u - 7x^3u' + 16x^2u = 0$$

$$8x^3u' + x^4u'' - 7x^3u' = 0 \Rightarrow xu'' + u' = 0.$$

$$\text{Let } w = u' \text{ then } w' = u'' \text{ and } xw' = -w \Rightarrow \frac{dw}{w} = -\frac{1}{x}dx \Rightarrow \ln(w) = -\ln(x) \Rightarrow w = \frac{1}{x}$$

$$u' = \frac{1}{x} \Rightarrow u(x) = \ln(x) \text{ and } y_2(x) = x^4 \ln(x).$$

So the general solution is  $y(x) = C_1x^4 + C_2x^4 \ln(x)$ .

**Q.2:** Solve the initial value problem  $y'' + 2y' + 5y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ .

**Sol:** Auxiliary equation is  $m^2 + 2m + 5 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 + 2i$

$$y(x) = e^{-x} (A \cos(2x) + B \sin(2x))$$

$$y'(x) = -e^{-x} (A \cos(2x) + B \sin(2x)) + e^{-x} (-2A \sin(2x) + 2B \cos(2x))$$

$$y(0) = 1 \Rightarrow A = 1 \text{ and } y'(0) = 0 \Rightarrow 0 = -A + 2B \Rightarrow B = \frac{1}{2}A = \frac{1}{2}$$

$$y(x) = e^{-x} \left( \cos(2x) + \frac{1}{2} \sin(2x) \right).$$

**Q.3:** Solve  $y'' + 3y' + 2y = 4x + 2$  using Undetermined coefficients (Annihilator Approach).

**Sol:** Auxiliary equation of the corresponding equation is  $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$$y_c(x) = C_1 e^{-x} + C_2 e^{-2x}$$

$$(D^2 + 3D + 2)y = 4x + 2 \Rightarrow D^2(D^2 + 3D + 2)y = D^2(4x + 2) = 0.$$

Auxiliary equation of the last fourth order homogeneous equation is

$$m^2(m^2 + 3m + 2) = 0 \Rightarrow m = 0, 0, -1, -2 \Rightarrow y(x) = C_1 + C_2 x + C_3 e^{-x} + C_4 e^{-2x}$$

$$y_p(x) = C_1 + C_2 x.$$

**Q.4:** If  $m_1 = -3$  and  $m_2 = 3 - i$  are roots of a cubic auxiliary equation of a third order differential equation, find the corresponding homogeneous linear differential equation.

**Sol:** If  $m_2 = 3 - i$  is a complex root then its conjugate  $m_3 = 3 + i$  is also a root.

$$(m + 3)(m - 3 + i)(m - 3 - i) = 0 \Rightarrow (m + 3)((m - 3)^2 + 1) = 0$$

$$\Rightarrow m^3 - 3m^2 - 8m + 30 = 0$$

The corresponding equation is  $y''' - 3y'' - 8y' + 30y = 0$ .