Solution Math 202 Quiz 3 (A)

Q.1: If $y_1 = x^4$ is a solution of the differential equation $x^2y'' - 7xy' + 16y = 0$, find a second solution and write the general solution.

Sol: First we write $x^2y'' - 7xy' + 16y = 0$ as $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0$. Then $P(x) = -\frac{7}{x}$

$$y_2(x) = y_1(x) \int \frac{e^{\int -P(x)dx}}{(y_1(x))^2} dx = x^4 \int \frac{e^{\int \frac{7}{x}dx}}{x^8} dx = x^4 \int \frac{x^7}{x^8} dx = x^4 \ln(x).$$

So the general solution is $y(x) = C_1x^4 + C_2x^4 \ln(x)$

OR: Let $y(x) = u(x)\, y_1(x) = x^4u$, then $y' = 4x^3u + x^4u'$ and $y'' = 12x^2u + 4x^3u' + 4x^3u' + x^4u''$

$$12x^2u + 8x^3u' + x^4u'' - 28x^2u - 7x^3u' + 16x^2u = 0$$

$$8x^3u' + x^4u'' - 7x^3u' = 0 \ \Rightarrow xu'' + u' = 0.$$

Let $w = u'$ then $w' = u''$ and $xw' = -w \ \Rightarrow \frac{dw}{w} = -\frac{1}{x}dx \ \Rightarrow \ln(w) = -\ln(x) \ \Rightarrow w = \frac{1}{x}$

$u' = \frac{1}{x} \ \Rightarrow u(x) = \ln(x)$ and $y_2(x) = x^4 \ln(x)$.

So the general solution is $y(x) = C_1x^4 + C_2x^4 \ln(x)$.

Q.2: Solve the initial value problem $y'' + 2y' + 5y = 0,\ y(0) = 1,\ y'(0) = 0$.

Sol: Auxiliary equation is $m^2 + 2m + 5 = 0 \ \Rightarrow m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 + 2i$

$$y(x) = e^{-x}(A\cos(2x) + B\sin(2x))$$

$$y'(x) = -e^{-x}(A\cos(2x) + B\sin(2x)) + e^{-x}(-2A\sin(2x) + 2B\cos(2x))$$

$$y(0) = 1 \ \Rightarrow A = 1 \text{ and } y'(0) = 0 \ \Rightarrow 0 = -A + 2B \ \Rightarrow B = \frac{1}{2}A = \frac{1}{2}$$

$$y(x) = e^{-x}\left(\cos(2x) + \frac{1}{2}\sin(2x)\right).$$

Q.3: Solve $y'' + 3y' + 2y = 4x + 2$ using Undetermined coefficients (Annihilator Approach).

Sol: Auxiliary equation of the corresponding equation is $m^2 + 3m + 2 = 0 \ \Rightarrow m = -1, -2$

$y_c(x) = C_1 e^{-x} + C_2 e^{-2x}$

$(D^2 + 3D + 2)\, y = 4x + 2 \ \Rightarrow D^2 (D^2 + 3D + 2)\, y = D^2 (4x + 2) = 0.$

Auxiliary equation of the last fourth order homogeneous equation is

$m^2 (m^2 + 3m + 2) = 0 \Rightarrow m = 0, 0, -1, -2 \ \Rightarrow y(x) = C_1 + C_2 x + C_3 e^{-x} + C_4 e^{-2x}$

$y_p(x) = C_1 + C_2 x.$

Q.4: If $m_1 = -3$ and $m_2 = 3 - i$ are roots of a cubic auxiliary equation of a third order differential equation, find the corresponding homogeneous linear differetial equation.

Sol: If $m_2 = 3 - i$ is a complex root then its conjugate $m_3 = 3 + i$ is also a root.

$(m + 3)(m - 3 + i)(m - 3 - i) = 0 \ \Rightarrow (m + 3)\left((m - 3)^2 + 1\right) = 0$

$\Rightarrow m^3 - 3m^2 - 8m + 30 = 0$

The corresponding equation is $y''' - 3y'' - 8y' + 30y = 0.$