Solution Math 202-091 Quiz 2 (B)

Q.1: Solve the differential equation $(\sin x + 3x^2y^2)\,dx + (2x^3y + \cos y)\,dy = 0.$

Sol: $M(x,y) = \sin x + 3x^2y^2$ and $N(x,y) = 2x^3y + \cos y$

$M_y = 6x^2y = N_x$, the equation is exact.

$$\frac{\partial f}{\partial x} = M(x,y) = \sin x + 3x^2y^2 \;\Rightarrow f(x,y) = -\cos x + x^3y^2 + g(y)$$

$$\frac{\partial f}{\partial y} = 2x^3y + g'(y) = N(x,y) = 2x^3y + \cos y \;\Rightarrow g'(y) = \cos y \;\Rightarrow g(y) = \sin y + C$$

Thus one parameter family of solutions is $-\cos x + x^3y^2 + \sin y + C = 0.$

Q.2: Solve the initial value problem $\sin x\frac{dy}{dx} + (1+\cos x)\,y = \frac{1}{1-\cos x},\; y\left(\frac{\pi}{2}\right) = 1.$(initial condition is changed)

Sol: Given equation can be written as $\frac{dy}{dx} + \frac{(1+\cos x)}{\sin x}y = \frac{1}{\sin x\,(1-\cos x)}$

which is a linear equation with $P(x) = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \csc x - \cot x$

$$IF = e^{\int(\csc x - \cot x)dx} = e^{\ln(\csc x - \cot x) + \ln \sin x} = e^{\ln\left(\frac{1-\cos x}{\sin x}\right)\sin x} = 1 - \cos x$$

$$\frac{d}{dx}\left((1-\cos x)\,y\right) = \csc x \;\Rightarrow (1-\cos x)y = \ln(\csc x - \cot x) + C$$

$$y\left(\frac{\pi}{2}\right) = 1 \;\Rightarrow 1 = \ln(1-0) + C \;\Rightarrow C = 1 \;\Rightarrow\; y = \frac{\ln(\csc x - \cot x) + 1}{1-\cos x}$$ is the solution.

Q.3: Transform the equation into a separable equation $(2xy + 3x^2)\,dx + 2y^2dy = 0.$

Sol: Given equation is a homogeneous equation.

Putting $y = ux$ and $dy = udx + xdu$, we get $(2x^2u + 3x^2)\,dx + 2x^2u^2\,(udx + xdu) = 0$

$(2u + 3 + 2u^3)\,dx + 2u^2xdu = 0 \;\Rightarrow \frac{dx}{x} = \frac{-2u^2}{2u^3 + 2u + 3}du$, a separable equation.

Q.4: Transform the equation into a linear equation $(-2x^2y + 3xy^4)\,dx = 2x^3dy.$

Sol: We can write $\frac{dy}{dx} = -\frac{1}{x}y + \frac{3}{2x^2}y^4 \;\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = \frac{3}{2x^2}y^4$, a Bernoulli's equation with $n = 4$.

Put $u = y^{1-4} = y^{-3}$ or $y = u^{-\frac{1}{3}} \;\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-1}{3}u^{-\frac{4}{3}}\frac{du}{dx}$

$$\frac{-1}{3}u^{-\frac{4}{3}}\frac{du}{dx} + \frac{1}{x}u^{-\frac{1}{3}} = \frac{3}{2x^2}u^{-\frac{4}{3}} \;\Rightarrow\; \frac{du}{dx} - \frac{3}{x}u = \frac{-9}{x^2}$$ a linear equation.