Solution Math 202 Quiz 2 (B)

Q.1: Solve the initial value problem $ydx - 3\left(x+y^4\right)dy = 0$, $y(1) = 1$. (Hint: write as a linear equation)

Sol: Write DE as a linear equation in x and $\frac{dx}{dy}$ as $\frac{dx}{dy} - \frac{3}{y}x = 3y^3 \quad (*)$

The integrating factor is $IF = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{y^{-3}} = y^{-3} = \frac{1}{y^3}$

Multiply $(*)$ by IF, $\frac{1}{y^3}\frac{dx}{dy} - \frac{3}{y^4}x = 3 \ \Rightarrow \frac{d}{dy}\left(\frac{1}{y^3}x\right) = 3 \ \Rightarrow \frac{1}{y^3}x = 3y + C$

$x = 3y^4 + Cy^3$. $y(1) = 1 \ \Rightarrow C = -2$. So the solution is $x = 3y^4 - 2y^3$.

Q.2: Solve the differential equation $\left(1+\frac{2}{x}\right)\sin(y)\,dx + \cos(y)\,dy = 0$ by transforming into an EXACT equation.

Sol: $M(x,y) = \left(1+\frac{2}{x}\right)\sin(y)$, $N(x,y) = \cos(y)$ and $M_y = \left(1+\frac{2}{x}\right)\cos(y)$, $N_x = 0$

$$\frac{M_y - N_x}{N} = \frac{\left(1+\frac{2}{x}\right)\cos(y) - 0}{\cos(y)} = 1 + \frac{2}{x} \text{ a function of } x \text{ alone.}$$

Integrating factor is $\mu(x) = e^{\int\left(1+\frac{2}{x}\right)dx} = e^{x+2\ln x} = e^x e^{\ln x^2} = x^2e^x$

Multiplying given DE by x^2e^x we get $e^x\left(x^2+2x\right)\sin(y)\,dx + x^2e^x\cos(y)\,dy = 0$, which is EXACT.

$\frac{\partial f}{\partial y} = \bar{N}(x,y) = x^2e^x\cos(y) \ \Rightarrow f(x,y) = \int x^2e^x\cos(y)\,dy + g(x) = x^2e^x\sin(y) + g(x)$

$\frac{\partial f}{\partial x} = \left(x^2e^x + 2xe^x\right)\sin(y) + g'(x) = \left(x^2+2x\right)e^x\sin(y) + g'(x) = \bar{M}(x,y) = e^x\left(x^2+2x\right)\sin(y)$

$g'(x) = 0 \ \Rightarrow g(x) = C$.

$f(x,y) = x^2e^x\sin(y) + C = 0$ OR $x^2e^x\sin(y) = C$.

OR $$\frac{N_x - M_y}{M} = \frac{0 - \left(1+\frac{2}{x}\right)\cos(y)}{\left(1+\frac{2}{x}\right)\sin(y)} = -\cot(y) \text{ a function of } y \text{ alone.}$$

Integrating factor is $\mu(y) = e^{\int -\cot(y)dy} = e^{-\ln(\sin y)} = \frac{1}{\sin y}$

Multiplying given DE by $\frac{1}{\sin y}$ we get $\left(1+\frac{2}{x}\right)dx + \cot(y)\,dy = 0$

This is separable equation and its solution is

$$(x + 2\ln x) + \ln(\sin y) = \ln C \quad \Rightarrow (\ln e^x + \ln x^2) + \ln(\sin y) = \ln C \quad \Rightarrow x^2 e^x \sin(y) = C$$

Q.3: Solve the differential equation $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$ by transforming into a separable equation.

Sol: This is a Homogeneous DE of degree 2.

Let $y = ux$ and $dy = xdu + udx$

$$(x^2 + x^2u^2)\,dx + (x^2 - x^2u)\,(xdu + udx) = 0$$

$$x^2(1 + u^2)\,dx + x^2(1 - u)\,(xdu + udx) = 0 \qquad \text{Divide by } x^2$$

$$(1 + u^2)\,dx + x(1 - u)\,du + u(1 - u)\,dx = 0 \;\Rightarrow (1 + u^2 + u - u^2)\,dx + x(1 - u)\,du = 0$$

$$\frac{u - 1}{(u + 1)}du = \frac{1}{x}dx \;\Rightarrow \left(1 - \frac{2}{u + 1}\right)du = \frac{1}{x}dx$$

$$u - 2\ln|u + 1| = \ln|x| + \ln C$$

$$\ln e^u - \ln(u + 1)^2 = \ln|x| + \ln C$$

$$\frac{e^u}{(u + 1)^2} = Cx \quad \Rightarrow e^{\frac{y}{x}} = Cx\left(\frac{y}{x} + 1\right)^2 = C\frac{(x + y)^2}{x}$$

Q.4: Solve the Bernoulli differential equation $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$ by transforming into a linear equation.

Sol: $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x}y^{-2}$ This is a Bernoulli equation with $n = -2$.

Put $u = y^{1-n} = y^3$ OR $y = u^{\frac{1}{3}}$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{3}u^{-\frac{2}{3}}\dfrac{du}{dx}$

$\dfrac{1}{3}u^{-\frac{2}{3}}\dfrac{du}{dx} + \dfrac{1}{x}u^{\frac{1}{3}} = \dfrac{1}{x}u^{-\frac{2}{3}} \;\Rightarrow \dfrac{du}{dx} + \dfrac{3}{x}u = \dfrac{3}{x}$ This is a linear equation in u and $\dfrac{du}{dx}$.

$$IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\frac{d}{dx}(x^3 u) = 3x^2 \;\Rightarrow x^3 u = x^3 + C \;\Rightarrow y^3 = 1 + Cx^{-3}.$$