Math 202-102 Sec: 21 Solution Quiz 2 (A)

Q.1: Solve the initial value problem $(x+1)\dfrac{dy}{dx} + y = \ln(x)$, $y(1) = 10$.

Sol. We can write the given equation as $\dfrac{d}{dx}\left[(x+1)y\right] = \ln(x)$

Which $\Rightarrow (x+1)y = \int \ln(x)dx + c$, $\Rightarrow (x+1)y = x\ln(x) - x + c$ or $y = \dfrac{x\ln(x) - x + c}{x+1}$

$y(1) = 10 \Rightarrow 10 = \dfrac{0 - 1 + c}{2} \Rightarrow c = 21$ and the solution is $y = \dfrac{x\ln(x) - x + 21}{x+1}$.

Q.2: Solve the differential equation $(2y^2 + 3x)dx + 2xydy = 0$ by making it exact.

Sol. $M(x,y) = 2y^2 + 3x$ and $N(x,y) = 2xy$. $\dfrac{\partial M}{\partial y} = 4y$ and $\dfrac{\partial N}{\partial x} = 2y$. So the equation is not exact.

$\dfrac{M_y - N_x}{N} = \dfrac{4y - 2y}{2xy} = \dfrac{1}{x}$ is a function of x alone.

$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$ is the integrating factor.

Multiplying by integration factor, we get $(2xy^2 + 3x^2)dx + 2x^2ydy = 0$

$\overline{M}(x,y) = 2xy^2 + 3x^2$ and $\overline{N}(x,y) = 2x^2y$. $\dfrac{\partial \overline{M}}{\partial y} = 4xy$ and $\dfrac{\partial \overline{N}}{\partial x} = 4xy$. So the equation is exact.

There exists a function $f(x,y)$ such that $\dfrac{\partial f}{\partial x} = \overline{M}$ and $\dfrac{\partial f}{\partial y} = \overline{N}$

$\dfrac{\partial f}{\partial x} = \overline{M} = 2xy^2 + 3x^2, \ \Rightarrow f(x,y) = x^2y^2 + x^3 + g(y)$

$\dfrac{\partial f}{\partial y} = 2x^2y + g'(y) = \overline{N} = 2x^2y \ \Rightarrow g'(y) = 0 \ \Rightarrow g(y) = c$

So $f(x,y) = x^2y^2 + x^3 + c$

Q.3: (a.) Use appropriate substitution to transform $(x^2 + y^2)dx + (x^2 - xy)dy = 0$ into a separable equation. Do not solve the new equation.

Sol. Given equation is a homogeneous equation of degree 2. Let $y = ux$, then $dy = udx + xdu$

$(x^2 + y^2)dx + (x^2 - xy)dy = 0 \ \Rightarrow (x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) = 0$

$x^2dx + u^2x^2dx + x^2udx - u^2x^2dx + x^3du - ux^3du = 0 \ \Rightarrow x^2(1+u)dx + x^3(1-u)du = 0$

$x^2(1+u)dx + x^3(1-u)du = 0 \ \Rightarrow x^2(1+u)dx = -x^3(1-u)du \ \Rightarrow \dfrac{1}{x}dx = \dfrac{u-1}{u+1}du$

Which is a separable equation.

(b.) Use appropriate substitution to transform $x\dfrac{dy}{dx} + y = x^2y^2$ into a linear equation. Do not solve the new equation.

Sol. $x\dfrac{dy}{dx} + y = x^2y^2 \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x} = xy^2$ which is a Bernoulli equation with $n = 2$.

Let $u = y^{1-2} = y^{-1}$ or $y = u^{-1}$ and $\dfrac{dy}{dx} = -u^{-2}\dfrac{du}{dx}$

So $\dfrac{dy}{dx} + \dfrac{y}{x} = xy^2 \quad \Rightarrow -u^{-2}\dfrac{du}{dx} + \dfrac{u^{-1}}{x} = xu^{-2}$

Multiplying by $-u^2$ we get $\dfrac{du}{dx} - \dfrac{u}{x} = -x$ which is a linear equation.