Solution Math 202 Quiz 2 (A)

Q.1: Solve the initial value problem $ydx - 2(x + y^3)\,dy = 0, \quad y(1) = 1$. (Hint: write as a linear equation)

Sol: Write DE as a linear equation in x and $\frac{dx}{dy}$ as $\frac{dx}{dy} - \frac{2}{y}x = 2y^2 \quad (*)$

The integrating factor is $IF = e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = e^{y^{-2}} = y^{-2} = \frac{1}{y^2}$

Multiply $(*)$ by IF, $\frac{1}{y^2}\frac{dx}{dy} - \frac{2}{y^3}x = 2 \;\Rightarrow\; \frac{d}{dy}\left(\frac{1}{y^2}x\right) = 2 \;\Rightarrow\; \frac{1}{y^2}x = 2y + C$

$x = 2y^3 + Cy^2$. $y(1) = 1 \;\Rightarrow C = -1$. So the solution is $x = 2y^3 - y^2$.

Q.2: Solve the differential equation $\cos(x)\,dx + \left(1 + \frac{2}{y}\right)\sin(x)\,dy = 0$ by transforming into an EXACT equation.

Sol: $M(x,y) = \cos(x)$, $N(x,y) = \left(1 + \frac{2}{y}\right)\sin(x)$ and $M_y = 0,\; N_x = \left(1 + \frac{2}{y}\right)\cos(x)$

$$\frac{N_x - M_y}{M} = \frac{\left(1 + \frac{2}{y}\right)\cos(x) - 0}{\cos(x)} = 1 + \frac{2}{y} \text{ a function of } y \text{ alone.}$$

Integrating factor is $\mu(y) = e^{\int\left(1+\frac{2}{y}\right)dy} = e^{y + 2\ln y} = e^y e^{\ln y^2} = y^2 e^y$

Multiplying given DE by y^2e^y we get $y^2e^y\cos(x)\,dx + (y^2 + 2y)\,e^y\sin(x)\,dy = 0$ which is EXACT.

$$\frac{\partial f}{\partial x} = \bar{M}(x,y) = y^2e^y\cos(x) \;\Rightarrow f(x,y) = \int y^2e^y\cos(x)\,dx + g(y) = y^2e^y\sin(x) + g(y)$$

$$\frac{\partial f}{\partial y} = (y^2e^y + 2ye^y)\sin(x) + g'(y) = (y^2 + 2y)\,e^y\sin(x) + g'(y) = \bar{N}(x,y) = (y^2 + 2y)\,e^y\sin(x)$$

$g'(y) = 0 \;\Rightarrow g(y) = C$.

$f(x,y) = y^2e^y\sin(x) + C = 0$ OR $y^2e^y\sin(x) = C$.

$$\text{OR } \frac{M_y - N_x}{N} = \frac{0 - \left(1 + \frac{2}{y}\right)\cos(x)}{\left(1 + \frac{2}{y}\right)\sin(x)} = -\cot(x) \text{ a function of } x \text{ alone.}$$

Integrating factor is $\mu(x) = e^{\int -\cot(x)dx} = e^{-\ln(\sin x)} = \frac{1}{\sin x}$

Multiplying given DE by $\frac{1}{\sin x}$ we get $\cot(x)\,dx + \left(1+\frac{2}{y}\right)dy = 0$

This is separable equation and its solution is

$$\ln(\sin x) + (y + 2\ln y) = \ln C \quad \Rightarrow \ln(\sin x) + (\ln e^y + \ln y^2) = \ln C \quad \Rightarrow y^2 e^y \sin(x) = C$$

Q.3: Solve the differential equation $(y^2+xy)\,dx + x^2 dy = 0$ by transforming into a separable equation.

Sol: This is a Homogeneous DE of degree 2.

Let $y = ux$ and $dy = xdu + udx$

Then $(u^2x^2 + x^2u)\,dx + x^2(xdu + udx) = 0$

$x^2(u^2+u)\,dx + x^2(xdu+udx) = 0$ Divide by x^2

$$(u^2+2u)\,dx + xdu = 0 \;\Rightarrow \frac{1}{x}dx = \frac{-1}{u(u+2)}du = \frac{1}{2}\left(\frac{1}{u+2} - \frac{1}{u}\right)du$$

$$\ln|x| = \frac{1}{2}\left[\ln|u+2| - \ln|u|\right] + \ln C$$

$$\ln|x| = \frac{1}{2}\left[\ln\left|\frac{u+2}{u}\right|\right] + \ln C = \frac{1}{2}\left[\ln\left|1+\frac{2}{u}\right|\right] + \ln C$$

$$x = C\sqrt{\left(1+\frac{2}{u}\right)} \;\Rightarrow x = C\sqrt{\left(1+\frac{2x}{y}\right)}.$$

Q.4: Solve the Bernoulli differential equation $x\frac{dy}{dx} + y = x^2y^2$ by transforming into a linear equation.

Sol: $\frac{dy}{dx} + \frac{y}{x} = xy^2$ This is a Bernoulli equation with $n = 2$.

Put $u = y^{1-n} = y^{-1}$ OR $y = u^{-1} = \frac{1}{u}$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-1}{u^2}\frac{du}{dx}$

$\frac{-1}{u^2}\frac{du}{dx} + \frac{1}{xu} = \frac{x}{u^2} \;\Rightarrow \frac{du}{dx} - \frac{u}{x} = -x$ This is a linear equation in u and $\frac{du}{dx}$.

$$IF = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = -1 \;\Rightarrow \frac{1}{x}u = -x + C \;\Rightarrow \frac{1}{xy} = -x + C$$

$$y = \frac{1}{-x^2 + Cx}$$