Solution Math 202 Quiz 1 (B)

Q.1: Verify that $y = -x\cos(x) + \sin(x)\ln(\sin(x))$ is a solution of $y'' + y = \csc(x)$.

Sol: $y = -x\cos(x) + \sin(x)\ln(\sin(x))$,

$$y' = -\cos x + x\sin x + \cos x \ln(\sin x) + \sin x \frac{\cos x}{\sin x} = x\sin x + \cos x \ln(\sin x)$$

$$y'' = \sin x + x\cos x - \sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x}$$

$$y'' + y = \sin x + x\cos x - \sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x} - x\cos(x) + \sin(x)\ln(\sin(x))$$

$$= \sin x + \frac{\cos^2 x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x} = \csc x$$

Q.2: Determine a region of the $xy-plane$ for which the differential equation $\left(\sqrt{y^2-4}\right) y' = (x^2-4)(y^2-4)$ would have a unique solution containing the point (x_o, y_o).

Sol: $y' = (x^2-4)\dfrac{(y^2-4)}{\sqrt{y^2-4}} = (x^2-4)\sqrt{y^2-4}$

$f(x,y) = (x^2-4)\sqrt{y^2-4}$ and $\dfrac{\partial f}{\partial y} = (x^2-4)\dfrac{y}{\sqrt{y^2-4}}$

$f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on the two planes $y > 2$ and $y < 2$.

Q.3: Solve the differential equation $\sqrt{1-y^2}dx - \sqrt{1-x^2}dy = 0$ with $y\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2}$

Sol: $\dfrac{1}{\sqrt{1-y^2}}dy = \dfrac{1}{\sqrt{1-x^2}}dx$

$\sin^{-1} y = \sin^{-1} x + C$

$$y\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2} \Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + C \Rightarrow \frac{\pi}{6} = \frac{\pi}{3} + C \Rightarrow C = -\frac{\pi}{6}$$

$$y = \sin\left(\sin^{-1} x + C\right) = \sin\left(\sin^{-1} x - \frac{\pi}{6}\right)$$

$$= \sin\left(\sin^{-1} x\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\sin^{-1} x\right)\sin\left(\frac{\pi}{6}\right)$$

$$= x\,\frac{\sqrt{3}}{2} - \sqrt{1-x^2}\,\frac{1}{2} = \frac{\sqrt{3}x - \sqrt{1-x^2}}{2}.$$