Solution Math 202 Quiz 1 (A)

Q.1: Verify that $y = x\sin(x) + \cos(x)\ln(\cos(x))$ is a solution of $y'' + y = \sec(x)$.

Sol: $y = x\sin x + \cos x \ln(\cos x)$,

$$y' = \sin x + x\cos x - \sin x\ln(\cos x) - \cos x\frac{\sin x}{\cos x} = x\cos x - \sin x \ln(\cos x)$$

$$y'' = \cos x - x\sin x - \cos x\ln(\cos x) + \sin x\frac{\sin x}{\cos x}$$

$$y'' + y = \cos x - x\sin x - \cos x\ln(\cos x) + \sin x\frac{\sin x}{\cos x} + x\sin(x) + \cos(x)\ln(\cos(x))$$

$$= \cos x + \frac{\sin^2 x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos x} = \sec x$$

Q.2: Determine a region of the $xy - plane$ for which the differential equation $\left(\sqrt{y^2-9}\right)y' = (x^2-1)(y^2-9)$ would have a unique solution containing the point (x_o, y_o).

Sol: $$y' = (x^2-1)\frac{(y^2-9)}{\sqrt{y^2-9}} = (x^2-1)\sqrt{y^2-9}$$

$f(x,y) = (x^2-1)\sqrt{y^2-9}$ and $\dfrac{\partial f}{\partial y} = (x^2-1)\dfrac{y}{\sqrt{y^2-9}}$

$f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on the two planes $y > 3$ and $y < 3$.

Q.3: Solve the differential equation $\sqrt{1-y^2}dx - \sqrt{1-x^2}dy = 0$ with $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$

Sol: $$\frac{1}{\sqrt{1-y^2}}dy = \frac{1}{\sqrt{1-x^2}}dx$$

$$\sin^{-1} y = \sin^{-1} x + C$$

$$y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2} \Rightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) + C \Rightarrow \frac{\pi}{3} = \frac{\pi}{6} + C \Rightarrow C = \frac{\pi}{6}$$

$$y = \sin\left(\sin^{-1} x + C\right) = \sin\left(\sin^{-1} x + \frac{\pi}{6}\right)$$

$$= \sin\left(\sin^{-1} x\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\sin^{-1} x\right)\sin\left(\frac{\pi}{6}\right)$$

$$= x\,\frac{\sqrt{3}}{2} + \sqrt{1-x^2}\,\frac{1}{2} = \frac{\sqrt{3}x + \sqrt{1-x^2}}{2}.$$