

Q.1: Solve the linear system $X' = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} -1-\lambda & 4 & 2 \\ 4 & -1-\lambda & -2 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 27\lambda + 4\lambda^2 - \lambda^3 - 90 = 0$

$27\lambda + 4\lambda^2 - \lambda^3 - 90 = 0$, Solution is: $\lambda = 3, -5, 6$.

For $\lambda = 3$, solve $(A - \lambda I)K = O \Rightarrow \begin{bmatrix} -4 & 4 & 2 & | & 0 \\ 4 & -4 & -2 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \Rightarrow K_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = -5$, solve $(A - \lambda I)K = O \Rightarrow \begin{bmatrix} 4 & 4 & 2 & | & 0 \\ 4 & 4 & -2 & | & 0 \\ 0 & 0 & 11 & | & 0 \end{bmatrix} \Rightarrow K_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 6$, solve $(A - \lambda I)K = O \Rightarrow \begin{bmatrix} -7 & 4 & 2 & | & 0 \\ 4 & -7 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} -7 & 4 & 2 & | & 0 \\ -3 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} -7 & 4 & 2 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \xrightarrow{7R_2+R_1} \begin{bmatrix} 0 & 11 & 2 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} K_3 = \begin{bmatrix} 2 \\ -2 \\ 11 \end{bmatrix}$

The three solutions are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$, $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 11 \end{bmatrix} e^{6t}$.

Q.2: Solve the linear system $X' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 3\lambda^2 - \lambda - \lambda^3 - 1 = 0$

$3\lambda^2 - \lambda - \lambda^3 - 1 = 0$, Solution is: $\lambda = 1, 1 - \sqrt{2}, 1 + \sqrt{2}$

For $\lambda = 1$, solve $(A - \lambda I)K = O \Rightarrow \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 2 & 1 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix} \Rightarrow K_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 1 + \sqrt{2}$, solve $(A - \lambda I)K = O \Rightarrow \begin{bmatrix} -\sqrt{2} & 0 & 0 & | & 0 \\ 2 & 1 - \sqrt{2} & 1 & | & 0 \\ 0 & 1 & -1 - \sqrt{2} & | & 0 \end{bmatrix}$

$$\Rightarrow K_2 = \begin{bmatrix} 0 \\ 1 + \sqrt{2} \\ 1 \end{bmatrix}$$

$$\text{For } \lambda = 1 - \sqrt{2}, \text{ solve } (A - \lambda I)K = O \Rightarrow \left[\begin{array}{ccc|c} \sqrt{2} & 0 & 0 & 0 \\ 2 & 1 + \sqrt{2} & 1 & 0 \\ 0 & 1 & -1 + \sqrt{2} & 0 \end{array} \right]$$

$$\Rightarrow K_2 = \begin{bmatrix} 0 \\ 1 - \sqrt{2} \\ 1 \end{bmatrix}$$

$$\text{The three solutions are } X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^t, X_2 = \begin{bmatrix} 0 \\ 1 + \sqrt{2} \\ 1 \end{bmatrix} e^{(1+\sqrt{2})t}, X_3 = \begin{bmatrix} 0 \\ 1 - \sqrt{2} \\ 1 \end{bmatrix} e^{(1-\sqrt{2})t}.$$

Q.3: Solve the linear system $X' = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} 1 - \lambda & -1 & 2 \\ -1 & 1 - \lambda & 0 \\ -1 & 0 & 1 - \lambda \end{vmatrix} = 3\lambda^2 - 4\lambda - \lambda^3 + 2 = 0$

$$3\lambda^2 - 4\lambda - \lambda^3 + 2 = 0, \text{ Solution is: } \lambda = 1, 1 - i, 1 + i$$

$$\text{For } \lambda = 1, \text{ solve } (A - \lambda I)K = O \Rightarrow \left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right] \Rightarrow K_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{For } \lambda = 1 + i, \text{ solve } (A - \lambda I)K = O \Rightarrow \left[\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array} \right] \xrightarrow[-R_2+R_3]{-iR_2+R_1} \left[\begin{array}{ccc|c} 0 & -2 & 2 & 0 \\ -1 & -i & 0 & 0 \\ 0 & i & -i & 0 \end{array} \right]$$

$$K_2 = \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix}$$

$$B_1 = \text{Re}(K_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } B_2 = \text{Im}(K_2) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{The three solutions are } X_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^t, X_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \sin t \right\} e^t = \begin{bmatrix} \sin t \\ \cos t \\ \cot s \end{bmatrix} e^t,$$

$$X_3 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \sin t \right\} e^t = \begin{bmatrix} -\cos t \\ \sin t \\ \sin t \end{bmatrix} e^t.$$