Solution Math 202 Quiz 6 (Cancelled) (A)

Q.1: Solve the linear system $X' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} = 0$

$7\lambda - \lambda^3 + 6 = 0$, Solution is: $\lambda = -1, 3, -2$.

For $\lambda = -1$, solve $(A - \lambda I)\, K = O \ \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda = -2$, solve $(A - \lambda I)\, K = O \ \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \xrightarrow{-R_1+R2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$

$\Rightarrow K_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

For $\lambda = 3$, solve $(A - \lambda I)\, K = O \ \Rightarrow \left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_1 + R2} \left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$

$\Rightarrow K_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

The three solutions are $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t}$, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t}$, $X_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t}$.

Q.2: Solve the linear system $X' = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} X$.

Sol: Eigenvalues are given by $\begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \ \Rightarrow \lambda = 4, 4, 4.$

For $\lambda = 4$, solve $(A - \lambda I)\, K = O \ \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow K = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

To find vector P, solve $(A - \lambda I)\, P = K \ \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow P = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

To find vector Q, solve $(A-\lambda I)\,Q = P \;\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The three solutions are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{4t},\; X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{4t}$

$X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dfrac{t^2}{2} e^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{4t}.$

Q.3: Solve the linear system $X' = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{bmatrix} X.$

Sol: Eigenvalues are given by $\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 6-\lambda & 0 \\ -4 & 0 & 4-\lambda \end{vmatrix} = 14\lambda^2 - 68\lambda - \lambda^3 + 120 = 0$

$14\lambda^2 - 68\lambda - \lambda^3 + 120 = 0$, Solution is: $\lambda = 6, 4-2i, 4+2i$

For $\lambda = 6$, solve $(A-\lambda I)\,K = O \;\Rightarrow \left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & -2 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

For $\lambda = 4+2i$, solve $(A-\lambda I)\,K = O \;\Rightarrow \left[\begin{array}{ccc|c} -2i & 0 & 1 & 0 \\ 0 & 2-2i & 0 & 0 \\ -4 & 0 & -2i & 0 \end{array}\right] \Rightarrow K_2 = \begin{bmatrix} 1 \\ 0 \\ 2i \end{bmatrix}$

$B_1 = \mathrm{Re}\,(K_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $B_2 = \mathrm{Im}(K_2) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

The three solutions are $X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{6t},$

$$X_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \sin 2t \right\} e^{4t} = \begin{bmatrix} \cos 2t \\ 0 \\ -2\sin 2t \end{bmatrix} e^{4t},$$

$$X_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sin 2t \right\} e^{4t} = \begin{bmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{bmatrix} e^{4t}.$$