King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Solution of Math 202 Final Exam
The First Semester of 2009-2010 (091)

Time Allowed: 180 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ________________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		12
2		10
3		12
4		10
5		12
6		23
7		14
8		12
9		10
10		15
Total		130

Q.1: Given the following differential equation

$$\cos x \; dx + \left(1 + \frac{2}{y}\right) \sin x \; dy = 0.$$

(a) (2-points) Determine whether the differential equation is EXACT or not?

Sol: Here $M(x, y) = \cos x$ and $N(x, y) = \left(1 + \frac{2}{y}\right) \sin x$.

$M_y = 0$ and $N_x = \left(1 + \frac{2}{y}\right) \cos x$.

Since $M_y \neq N_x$, therefore the given equation is not exact.

(b) (4-points) Find integrating factor and make the differential equation EXACT.

Sol: $\dfrac{M_y - N_x}{N} = \dfrac{-\left(1 + \frac{2}{y}\right) \cos x}{\left(1 + \frac{2}{y}\right) \sin x} = -\cot x$, a function of x alone.

The integrating factor is $\mu(x) = e^{\int -\cot x \; dx} = e^{-\ln(\sin x)} = (\sin x)^{-1} = \dfrac{1}{\sin x}$.

$\mu(x) \cos x \; dx + \mu(x) \left(1 + \frac{2}{y}\right) \sin x \; dy = 0$

$\Rightarrow \cot x dx + \left(1 + \frac{2}{y}\right) dy = 0$, which is the required EXACT equation.

(c) (6-points) Solve the EXACT differential equation obtained in part (b).

Sol: $f(x, y) = \int M(x, y) \, dx + g(y) = \int \cot x dx + g(y) = \ln(\sin x) + g(y)$.

$\dfrac{\partial f}{\partial y} = g'(y) = N(x, y) = \left(1 + \frac{2}{y}\right) \Rightarrow g(y) = y + 2 \ln y$.

So $f(x, y) = \ln(\sin x) + y + 2 \ln y = C_1$.

$\ln(\sin x) + \ln e^y + \ln y^2 = \ln e^{C_1} = \ln C$

$\sin x \; e^y \; y^2 = C$ is the required solution.

Q.2: (10-points) Solve the differential equation using an appropriate substitution

$$\left(1 + x^2\right) \frac{dy}{dx} = xy \left(y^2 - 1\right).$$

Sol: Given equation can be written as $\dfrac{dy}{dx} + \dfrac{x}{1 + x^2} y = \dfrac{x}{1 + x^2} y^3$ which is a Bernoullie's equation.

Let $u = y^{1-3} = y^{-2}$ OR $y = u^{-\frac{1}{2}}$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \dfrac{du}{dx} = -\dfrac{1}{2} u^{-\frac{3}{2}} \dfrac{du}{dx}$

Substituting in the equation we get $-\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx}+\frac{x}{1+x^2}u^{-\frac{1}{2}}=\frac{x}{1+x^2}u^{-\frac{3}{2}}$

$$\Rightarrow \frac{du}{dx}-\frac{2x}{1+x^2}u=-\frac{2x}{1+x^2}\Rightarrow \frac{du}{dx}=\frac{2x}{1+x^2}(u-1)\Rightarrow \frac{du}{(u-1)}=\frac{2x}{1+x^2}dx$$

$$\Rightarrow \ln(u-1)=\ln(1+x^2)+\ln C \ \Rightarrow u=1+C(1+x^2) \text{ OR } y=\left[1+C(1+x^2)\right]^{-\frac{1}{2}}$$

Q.3: (12-points) Use annihilator approach to solve the differential equation

$$y''+2y'+y=\cos^2 x-\sin^2 x.$$

Sol: Auxiliary equation of the associated homogeneous equation is

$m^2+2m+1=0 \ \Rightarrow m=-1,-1$

The complementary function is $y_c=c_1e^{-x}+c_2xe^{-x}$.

Given equation can be written as $(D^2+2D+1)\,y=\cos 2x$

$\Rightarrow (D^2+2^2)(D^2+2D+1)\,y=(D^2+2^2)\cos 2x=0$

Auxiliary equation of this equation is

$(m^2+4)(m^2+2m+1)=0 \ \Rightarrow m=-1,-1,\pm 2i.$

$y=c_1e^{-x}+c_2xe^{-x}+c_3\cos 2x+c_4\sin 2x.$

Let $y_p=A\cos 2x+B\sin 2x.$

Then $y_p'=-2A\sin 2x+2B\cos 2x$ and $y_p''=-4A\cos 2x-4B\sin 2x$

$y_p''+2y_p'+y_p=\cos 2x$

$\Rightarrow -4A\cos 2x-4B\sin 2x-4A\sin 2x+4B\cos 2x+A\cos 2x+B\sin 2x=\cos 2x$

Comparing coefficients we get $-3A+4B=1$ and $-4A-3B=0 \ \Rightarrow A=\frac{-3}{25},\ B=\frac{4}{25}.$

The general solution is $y=y_c+y_p=c_1e^{-x}+c_2xe^{-x}-\frac{3}{25}\cos 2x+\frac{4}{25}\sin 2x.$

Q.4: (10-points) Use variation of parameters method to find particular solution y_p of the differential equation

$$y''+4y=\csc 2x.$$

Sol: Auxiliary equation of the associated homogeneous equation $m^2+4=0$ has roots $m=\pm 2i.$

The complementary function is $y_c = A\cos 2x + B\sin 2x$.

Let $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Then for variaton of parameters method

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2,$$

$$W_1 = \begin{vmatrix} 0 & \sin 2x \\ \csc 2x & 2\cos 2x \end{vmatrix} = -1,\ W_2 = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \csc 2x \end{vmatrix} = \cot 2x$$

$$u_1(x) = \int \frac{W_1}{W}dx = \int \frac{-1}{2}dx = -\frac{1}{2}x,\ u_2(x) = \int \frac{W_2}{W}dx = \int \frac{\cot 2x}{2}dx = \frac{1}{4}\ln(\sin 2x)$$

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) = -\frac{x\cos 2x}{2} + \frac{\sin 2x\ \ln(\sin 2x)}{4}.$$

Q.5: (12-points) Solve the following initial value problem

$$2x^2y'' - 3xy' + 3y = 2x^3, \qquad y(1) = 0,\ y'(1) = 1.$$

Sol: Auxiliary equation of the given Cauchy-Euler equation is

$$2m(m-1) - 3m + 3 = 0\ \Rightarrow 2m^2 - 5m + 3 = 0\ m = 1, \frac{3}{2}.$$

$y_c = c_1x + c_2x^{\frac{3}{2}}$. Let $y_1 = x$ and $y_2 = x^{\frac{3}{2}}$.

Then $W = \begin{vmatrix} x & x^{\frac{3}{2}} \\ 1 & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = \frac{1}{2}x^{\frac{3}{2}},\ W_1 = \begin{vmatrix} 0 & x^{\frac{3}{2}} \\ x & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = -x^{\frac{5}{2}},\ W_2 = \begin{vmatrix} x & 0 \\ 1 & x \end{vmatrix} = x^2$

$$u_1(x) = \int \frac{W_1}{W}dx = -2\int x\,dx = -x^2,\ u_2(x) = \int \frac{W_2}{W}dx = 2\int x^{\frac{1}{2}}dx = \frac{4}{3}x^{\frac{3}{2}}.$$

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) = -x^3 + \frac{4}{3}x^3 = \frac{1}{3}x^3.$$

The general solution is $y = y_c + y_p = c_1x + c_2x^{\frac{3}{2}} + \frac{1}{3}x^3$.

$$y' = c_1 + \frac{3}{2}c_2x^{\frac{1}{2}} + x^2.$$

$$y(1) = 0\ \Rightarrow c_1 + c_2 + \frac{1}{3} = 0 \text{ and } y'(1) = 1\ \Rightarrow c_1 + \frac{3}{2}c_2 + 1 = 1$$

$$\Rightarrow c_1 = -1 \text{ and } c_2 = \frac{2}{3}$$

$$y = -x + \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}x^3.$$

Q.6: Consider the differential equation

$$xy'' - xy' + y = 0.$$

(a) (2-points) Show that $x = 0$ is a regular singular point of the differential equation.

Sol: $xy'' - xy' + y = 0 \Rightarrow y'' - y' + \frac{1}{x}y = 0 \Rightarrow P(x) = -1,\ Q(x) = \frac{1}{x}$

$p(x) = xP(x) = -x$ and $q(x) = x^2 Q(x) = x$

Both $p(x)$ and $q(x)$ are analytic at $x = 0$. Therefore $x = 0$ is a regular singular piont.

(b) (3-points) Write the indicial equation for the differential equation and show that indicial roots r_1 and r_2 satisfy $r_1 - r_2 = 1$.

Sol: $p(x) = -x \Rightarrow a_0 = 0$ and $q(x) = x \Rightarrow b_0 = 0$.

The indicial equation is $r(r-1) + a_0 r + b_0 = 0 \Rightarrow r = 0, 1$.

Let $r_1 = 1$ and $r_2 = 0$. Then $r_1 - r_2 = 1 - 0 = 1$.

(c) (10-points) Find a series solution of the differential equation of the form $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$.

Sol: For $r_1 = 1$, Let $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1} = \sum_{n=0}^{\infty} c_n x^{n+1}$

Then $y' = \sum_{n=0}^{\infty} (n+1) c_n x^n$ and $y'' = \sum_{n=1}^{\infty} n(n+1) c_n x^{n-1}$.

$$xy'' - xy' + y = 0 \Rightarrow \sum_{n=1}^{\infty} n(n+1) c_n x^n - \sum_{n=0}^{\infty} (n+1) c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\Rightarrow \sum_{k=1}^{\infty} k(k+1) c_k x^k - \sum_{k=1}^{\infty} k c_{k-1} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$\Rightarrow \sum_{k=1}^{\infty} \left[k(k+1) c_k - (k-1) c_{k-1}\right] = 0$$

$$c_k = \frac{k-1}{k(k+1)} c_{k-1}, \quad k = 1, 2, 3, ...$$

$$c_1 = \frac{0}{2} c_0 = 0 = c_2 = c_3 = ...$$

So $y_1 = c_0 x$ is the solution with c_0 arbitrary.

(d) (8-points) Use the reduction of order formula to show that the differential equation has a second series solution of the form $y_2 = x\left(-\frac{1}{x} + \ln x + \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)\,n!}\right)$. $\left(\text{Hint: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$

Sol: The reduction of order formula $y_2 = y_1(x) \int \frac{e^{\int -P(x)dx}}{\left(y_{1(x)}\right)^2}$

$$\Rightarrow y_2 = x \int \frac{e^{\int dx}}{x^2} dx = x \int \frac{e^x}{x^2} dx = x \int \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!}}{x^2} dx$$

$$= x \int \sum_{n=0}^{\infty} \frac{x^{n-2}}{n!} dx = x \int \left[\frac{1}{x^2} + \frac{1}{x} + \sum_{n=2}^{\infty} \frac{x^{n-2}}{n!}\right] dx = x \left[-\frac{1}{x} + \ln x + \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)\,n!}\right].$$

Q.7: Consider the following linear system of differential equations $X' = AX$.

(a) (3-points) Write the characteristic equation corresponding to the coefficient matrix

$$A = \begin{bmatrix} -3+s & -3 \\ 3 & 3 \end{bmatrix},$$

with s a scalar parameter.

Sol: The charactristic equation is $\begin{vmatrix} -3+s-\lambda & -3 \\ 3 & 3-\lambda \end{vmatrix} = 0$

$-9 + 3\lambda + 3s - s\lambda - 3\lambda + \lambda^2 + 9 = 0$

$\lambda^2 - s\lambda + 3s = 0.$

(b) (3-points) Find values of s for which the matrix A has complex eigenvalues.

Sol: For $\lambda^2 - s\lambda + 3s = 0$, the discriminant is $D = s^2 - 12s = s(s-12)$.

$D < 0$ for $s \in (0, 12)$.

So A has complex eigenvalues for $s \in (0, 12)$.

(c) (8-points) Solve the linear system for $s = 6$.

Sol: For $s = 6$, $\lambda^2 - 6\lambda + 18 = 0$, Solution is:$\lambda = 3 - 3i, 3 + 3i$

For $\lambda = 3 + 3i$,solve $(A - \lambda I) K = O \quad \Rightarrow \left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right]$

$k_1 = ik_2 \Rightarrow K = \begin{bmatrix} i \\ 1 \end{bmatrix}$. So $B_1 = \operatorname{Re}(K) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $B_2 = \operatorname{Im}(K) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The two solutions are $X_1 = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 3t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 3t\right\} e^{3t} = \begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix} e^{3t}$

and $X_1 = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 3t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 3t\right\} e^{3t} = \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} e^{3t}$

The general solution is $X = c_1 X_1 + c_2 X_2 = c_1 \begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} e^{3t}$.

Q.8: Consider the following linear system

$$\frac{dx}{dt} = y + z$$
$$\frac{dy}{dt} = 2x + y$$
$$\frac{dz}{dt} = -z$$

(*a*) (10-points) Write the system into matrix form $X' = AX$ and find eigenvalues and corresponding eigenvectors of the matrix A.

Sol: Given system can be written as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or $X' = AX$, with $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Eigenvalues of A are given by

$$\begin{vmatrix} 0-\lambda & 1 & 1 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)\left(\lambda^2 - \lambda - 2\right) = (2-\lambda)(\lambda+1)^2 = 0$$

$\Rightarrow \lambda = 2, -1, -1.$

For $\lambda = 2$, solve $(A - \lambda I) K_1 = O \Rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \Rightarrow K_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

For $\lambda = -1$, solve $(A - \lambda I) K_2 = O \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow K_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

K_1 is the only linearly independent eigenvector corresponding to $\lambda = -1$.

(*b*) (2-points) Find the general solution of the system.

Sol: To find vector P, solve $(A - \lambda I) P = K_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 + R_1} \left[\begin{array}{ccc|c} 0 & 0 & 1 & -\frac{3}{2} \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$P = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{3}{2} \end{bmatrix}.$$

The three solutions are

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{2t},\ X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t},\ X_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{3}{2} \end{bmatrix} e^{-t}$$

The general solution is $X(t) = c_1X_1 + c_2X_2 + c_3X_3$.

Q.9: (10-points) Consider the nonhomogeneous system

$$X' = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix} X + \begin{bmatrix} -3 \\ 6 \end{bmatrix},$$

Let $\Phi(t) = \begin{bmatrix} e^{3t} & -e^{-3t} \\ 5e^{3t} & e^{-3t} \end{bmatrix}$ be the fundamental matrix of the associated homogeneous system. Use variation of parameters method to find particular solution X_p and write the general solution.

Sol: The fundamental matrix is $\Phi(t) = \begin{bmatrix} e^{3t} & -e^{-3t} \\ 5e^{3t} & e^{-3t} \end{bmatrix}$ and $\Phi^{-1} = \frac{1}{6}\begin{bmatrix} e^{-3t} & e^{-3t} \\ -5e^{3t} & e^{3t} \end{bmatrix}$.

$$U(t) = \int \Phi^{-1}F\,dt = \int \frac{1}{6}\begin{bmatrix} e^{-3t} & e^{-3t} \\ -5e^{3t} & e^{3t} \end{bmatrix}\begin{bmatrix} -3 \\ 6 \end{bmatrix} dt = \int \frac{1}{6}\begin{bmatrix} 3e^{-3t} \\ 21e^{3t} \end{bmatrix} dt = \frac{1}{6}\begin{bmatrix} -e^{-3t} \\ 7e^{3t} \end{bmatrix}$$

$$X_p = \Phi(t)\,U(t) = \frac{1}{6}\begin{bmatrix} e^{3t} & -e^{-3t} \\ 5e^{3t} & e^{-3t} \end{bmatrix}\begin{bmatrix} -e^{-3t} \\ 7e^{3t} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -\dfrac{4}{3} \\ \dfrac{1}{3} \end{bmatrix}$$

The general solution is $X(t) = X_c + X_p = c_1\begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -\dfrac{4}{3} \\ \dfrac{1}{3} \end{bmatrix}$.

Q.10: Let the matrix A be given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(*a*) (2-points) Find the matrix B such that $A = I + B$.

Sol: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = I + B$, where $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(*b*) (2-points) Show that $B^3 = O$, (the zero matrix)

Sol: $B^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$B^3 = B^2 B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

(c) (3-points) Using $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots$ and part (b) show that

$$A^n = I + nB + \frac{n(n-1)}{2}B^2, \ n = 0, 1, 2, 3, ...$$

Sol: $A^n = (I+B)^n = I + nB + \dfrac{n(n-1)}{2!}B^2 + \dfrac{n(n-1)(n-2)}{3!}B^3 + \cdots = I + nB + \dfrac{n(n-1)}{2!}B^2.$

(d) (8-points) Compute the matrix exponential e^{At} and write the solution of $X' = AX$.

Sol: $e^{At} = I + At + \dfrac{A^2t^2}{2!} + \dfrac{A^3t^3}{3!} + \dfrac{A^4t^4}{4!} + \cdots$

$$= I + (I+B)t + \frac{t^2}{2!}(I + 2B + B^2) + \frac{t^3}{3!}(I + 3B + 3B^2) + \frac{t^4}{4!}(I + 4B + 6B^2) + \cdots$$

$$= I\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) + Bt\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)$$

$$+ \frac{B^2t^2}{2}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

$$= Ie^t + Bte^t + \frac{B^2t^2}{2}e^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} e^t + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} te^t + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{t^2}{2}e^t$$

$$= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} + \begin{bmatrix} 0 & te^t & 0 \\ 0 & 0 & te^t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{t^2}{2}e^t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}.$$

Solution of $X' = AX$ is

$$X = e^{At}C = \begin{bmatrix} e^t & te^t & \frac{t^2}{2}e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \frac{t^2}{2}e^t \\ te^t \\ e^t \end{bmatrix}$$