Math 202 Exam 1 Solution

Prepared by Dr. Muhammad Yousuf

Q.1: (a) (2-points) Consider the differential equation

$$\frac{dy}{dx} = y^2 \qquad\qquad (A)$$

Verify that $y = \dfrac{-1}{x+c}$ is a one paramenter family of solutions.(A).

Sol: $y = \dfrac{-1}{x+c} \Rightarrow \dfrac{dy}{dx} = -1\left(-1\right)\left(x+c\right)^2 = \dfrac{1}{\left(x+c\right)^2} = y^2.$

(b) (2-points) Find solutions of of equation (A) which satisfies the initial condition $y\left(0\right) = 2.$

Sol: $y\left(0\right) = 2 \Rightarrow 2 = \dfrac{-1}{0+c} \Rightarrow c = -\dfrac{1}{2}$ and $y = \dfrac{-1}{x - \frac{1}{2}} = \dfrac{-2}{x-1}.$

(c) (2-points) Find, if any, singular solutions of (A) and determine the largest interval of existance of each singular solution obtained.

Sol: Singular solutions are given by $y^2 = 0 \Rightarrow y = 0$ and $\dfrac{dy}{dx} = 0.$
The largest interval of existance is $\left(-\infty, \infty\right).$

Q.2: (a) (3-points) Given that $x\left(t\right) = c_1 \cos\left(wt\right) + c_2 \sin\left(wt\right)$ is the general solution of the differential equation

$$\ddot{x} + wx = 0 \qquad on \quad \left(-\infty, \infty\right), \quad w \neq 0.$$

Find all solutions which satisfy the initial conditions $x\left(0\right) = 0$ and $\dot{x}\left(\frac{\pi}{2w}\right) = 0.$

Sol: $x(0 = 0 \Rightarrow 0 = c_1 \cos\left(0\right) + c_2 \sin\left(0\right) = c_1.1 \Rightarrow c_1 = 0.$

$\dot{x}\left(t\right) = -c_1 w \sin\left(wt\right) + c_2 w \cos\left(wt\right)$

$\dot{x}\left(\frac{\pi}{2w}\right) = 0 \Rightarrow 0 = 0 + c_2 w \cos\left(\frac{\pi}{2}\right) = c_2 w\left(0\right)$

Since $w \neq 0$, therefore $0 = 0.c_2$ which is true for any real value of c_2.

Thus the required solutions are $x\left(t\right) = c_2 \sin\left(wt\right)$ with c_2 arbitrary.

(b) (3-points) Given $y_1 = e^x$ and $y_2 = e^x \tan x$ are two solutions of the differentia equation

$$y'' - 2\left(1 + \tan x\right) y' + \left(1 + 2\tan x\right) y = 0.$$

Determine whether or not the set $\{y_1, y_2\}$ form a fundamental set of solutions on the interval $\left(0, \frac{\pi}{2}\right)$.

Sol: For $y_1 = e^x$, $y_1' = e^x$ and for $y_2 = e^x \tan x$, $y_2' = e^x \tan x + e^x \sec^2 x$

$$\text{Now } W = \begin{vmatrix} e^x & e^x \tan x \\ e^x & e^x \tan x + e^x \sec^2 x \end{vmatrix} = \frac{1}{\cos^2 x} e^{2x} \neq 0 \text{ for all } x \in \left(0, \frac{\pi}{2}\right).$$

Hence $\{y_1, y_2\}$ forms a fundamental set of solutions.

Q.3: (a) (3-points) Find a suitable substitution that transforms the differential equation

$$xydy + \left(4x^2 + y^2\right) dx = 0$$

to a separable equation. **Find the new equation, but do not find its solution.**

Sol: Given DE is a homogeneous equation of degree 2. Let $y = ux$, then $dy = xdu + udx$

Substitute in DE $\Rightarrow ux^2 (xdu + udx) + (4x^2 + u^2x^2) dx = 0$

$\Rightarrow x^2 [uxdu + (2u^2 + 4) dx] = 0 \ \Rightarrow uxdu = -(2u^2 + 4) dx$

$\Rightarrow \left(\frac{u}{2u^2 + 4}\right) du = \frac{-1}{x} dx$ which is separable.

(b) (2-points) Find a suitable substitution that transforms the differential equation

$$\left(\cot x - y^2 \sin^3 x\right) dx + \frac{1}{y} dy = 0$$

into a linear equation. **Find the new equation, but do not find its solution.**

Sol: Given DE can be written as $\frac{dy}{dx} + (\cot x) y = y^3 \sin^3 x$ which is a Bernoulli equation with $n = 3$.

Let $u = y^{1-3} = y^{-2} \Rightarrow y^2 = u^{-1} \ \Rightarrow y = u^{-\frac{1}{2}} \ \Rightarrow \frac{dy}{dx} = -\frac{1}{2} u^{-\frac{3}{2}} \frac{du}{dx}$

$-\frac{1}{2} u^{-\frac{3}{2}} \frac{du}{dx} + (\cot x) u^{-\frac{1}{2}} = u^{-\frac{3}{2}} \sin^3 x$

$\frac{du}{dx} - 2 (\cot x) u = -2 \sin^3 x$ which is a linear equation.

Q.4: (7-points) Solve the initial value problem $x \ln(x) \frac{dy}{dx} + \cos^2(y) = 1,\ y(e) = \frac{\pi}{4}$.

Sol: $x \ln(x) \frac{dy}{dx} + \cos^2(y) = 1$

$x \ln(x) \frac{dy}{dx} = 1 - \cos^2(y)$

$$x\ln(x)\frac{dy}{dx} = \sin^2(y)$$

$$\frac{1}{\sin^2(y)}dy = \frac{1}{x\ln(x)}dx$$

$$\csc^2(y)\,dy = \frac{1}{x\ln(x)}dx$$

$$-\cot(y) = \ln(\ln|x|) + C$$

$$y(e) = \frac{\pi}{4} \quad \Rightarrow -\cot\left(\frac{\pi}{4}\right) = \ln\ln(e) + C \quad \Rightarrow -1 = \ln(1) + C \quad \Rightarrow C = -1.$$

$$\cot(y) + \ln(\ln|x|) = 1.$$

Q.5: (6-points) Find the general solution of the differential equation

$$ydx = (x + y\ln y)\,dy.$$

Sol: Given DE is linear in x and $\dfrac{dx}{dy}$ because it can be written as $\dfrac{dx}{dy} - \dfrac{1}{y}x = \ln y$.

$$IF = e^{\int -\frac{1}{y}dy} = e^{-\ln y} = \frac{1}{y}.$$

$$\frac{d}{dy}\left(\frac{x}{y}\right) = \frac{\ln y}{y} \quad \Rightarrow \frac{x}{y} = \int \frac{\ln y}{y}dy = \frac{(\ln y)^2}{2} + C$$

$$x = \frac{y(\ln y)^2}{2} + Cy.$$

Q.6: (7-points) Solve the differential equation $(3x + 4y^2)\,dx + 4xydy = 0$ by transforming it into an exact equation.

Sol: $M(x,y) = 3x + 4y^2$ and $N(x,y) = 4xy$

$M_y = 8y$ and $N_x = 4y$

Since $M_y \neq N_x$, the given DE is not exact. So we need to find an integrating factor.

$$\frac{M_y - N_x}{N} = \frac{8y - 4y}{4xy} = \frac{4y}{4xy} = \frac{1}{x}, \text{ function of } x \text{ alone.}$$

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x.$$

Multiply given DE by x to make it exact.

$\left(3x^2+4xy^2\right)dx+2x^2ydy=0.$

$\mu\left(x\right)N\left(x,y\right)=\dfrac{\partial f}{\partial y}=4x^2y$

$f\left(x,y\right)=\int 4x^2ydy+h\left(x\right)$

$f\left(x,y\right)=2x^2y^2+h\left(x\right)$

$\dfrac{\partial f}{\partial x}=4xy^2+h'\left(x\right)=\mu(x)M\left(x,y\right)=3x^2+4xy^2$

$h'\left(x\right)=3x^2 \quad \Rightarrow h\left(x\right)=x^3+C$

Thus $f\left(x,y\right)=2x^2y^2+x^3+C=0$

OR $2x^2y^2+x^3=C.$

Q.7: (6-points) A glass of water initially at 50^oF is placed in a freezer. The freezer is kept at the constant temperature 30^oF. After one hour the temperature of the water in glass is 40^oF. Find the exact time needed for the temperature of the water to reach 32^oF after it is placed in the freezer.

Sol: Solve the IVP $\dfrac{dT}{dt}=k\left(T-T_m\right),\ T\left(0\right)=50$, with $T_m=30^oF$.

$T\left(t\right)=T_m+ce^{kt}=30+ce^{kt}.$

Using $T\left(0\right)=50$ we get $c=20$. So $T\left(t\right)=30+20e^{kt}$

Using $T\left(1\right)=40 \Rightarrow 40=30+20e^k \ \Rightarrow k=-\ln 2=\ln\left(\frac{1}{2}\right).$

$T\left(t\right)=30+20e^{t\ln\left(\frac{1}{2}\right)}=30+20e^{\ln\left(\frac{1}{2}\right)^t}=30+20\left(\dfrac{1}{2}\right)^t.$

Now $T\left(t\right)=32 \ \Rightarrow 32=30+20\left(\dfrac{1}{2}\right)^t \ \Rightarrow \left(\dfrac{1}{2}\right)^t=\dfrac{1}{10} \ \Rightarrow t=\dfrac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{2}\right)}=\dfrac{\ln 10}{\ln 2}.$