Math 201 Quiz 5 (B)

Name:..Serial#:........................Sec #....................................

Q.1: Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$. Also find the shortest distance of $(1, 2, 3)$ from the plane.

Sol: Let $P(x, y, z)$ be any point on the plane. Its distance from the point $(1, 2, 3)$ is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$.

From equation of the plane, we get $z = 4 - x + y$ and rewrite $d = \sqrt{(x-1)^2 + (y-2)^2 + (1-x+y)^2}$.

Define $f(x, y) = d^2 = (x-1)^2 + (y-2)^2 + (1-x+y)^2$. We need to minimize the function $f(x, y)$.

$f_x = 2(x-1) - 2(1-x+y) = 4x - 2y - 4$ and $f_y = 2(y-2) + 2(1-x+y) = 4y - 2x - 2$.

$f_x = 0$ and $f_y = 0 \Rightarrow x = \dfrac{5}{3},\ y = \dfrac{4}{3}$.

Now $f_{xx} = 4,\ f_{yy} = 4,\ f_{xy} = -2$, and discriminant $D = f_{xx}f_{yy} - (f_{xy})^2 = 16 - 4 = 12 > 0$. Also $f_{xx} = 4 > 0$.

$\Rightarrow f(x, y)$ is minimum at $\left(\dfrac{5}{3}, \dfrac{4}{3}\right)$, that is, d is minimum.

Also $z = 4 - x + y = 4 - \dfrac{5}{3} + \dfrac{4}{3} = \dfrac{11}{3} \Rightarrow \left(\dfrac{5}{3}, \dfrac{4}{3}, \dfrac{11}{3}\right)$ is on the plane $x - y + z = 4$ and is closest to the point $(1, 2, 3)$.

The shortest distance is $d = \sqrt{\left(\dfrac{5}{3} - 1\right)^2 + \left(\dfrac{4}{3} - 2\right)^2 + \left(\dfrac{11}{3} - 3\right)^2} = \dfrac{2}{3}\sqrt{3}$.

Q.2: Find the local maximum and local minimum values of the function $f(x, y) = 14 - 2x - 6y + x^2 + y^2$.

Sol: $f_x = 2x - 2,\ f_y = 2y - 6$ and $f_x = 0,\ f_y = 0 \Rightarrow x = 1,\ y = 3$.

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0$. Thus $D = f_{xx}f_{yy} - (f_{xy})^2 = 4 > 0$, and $f_{xx} = 2 > 0 \Rightarrow f$ ha a local minimum at $(1, 3)$.

Q.3: Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$.

Sol: Define $g(x, y) = x^2 + y^2$. Then $\nabla f = \lambda \nabla g \Rightarrow 2x = 2x\lambda$ and $4y = 2y\lambda$.

$2x = 2x\lambda \Rightarrow x(1-\lambda) = 0 \Rightarrow x = 0$ or $\lambda = 1$. For $x = 0,\ g(x, y) = 1 \Rightarrow y = \pm 1$. So the points are $(0, \pm 1)$.

$4y = 2y\lambda \Rightarrow y(2-\lambda) = 0 \Rightarrow y = 0$ or $\lambda = 2$. For $y = 0,\ g(x, y) = 1 \Rightarrow x = \pm 1$. So the points are $(\pm 1, 0)$.

Now $f(0, \pm 1) = 2$ and $f(\pm 1, 0) = 1 \Rightarrow$maximum value of f subject to the constraint $x^2 + y^2 = 1$ is 2 and minimum value is 1.