Solution Math 201 Quiz 3

(A)

Q.1: Find the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{x^2+2y^2}$ if exist, or show that limit does not exist.

Sol: Let $(x,y)\to(0,0)$ through $y-axis$, that is $x=0$ and $y\to 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{x^2+2y^2} = \lim_{y\to 0}\frac{\sin^2 y}{2y^2} = \frac{1}{2}\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2 = \frac{1}{2}.$$

Now let $(x,y)\to(0,0)$ through $x-axis$, that is $y=0$ and $x\to 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{x^2+2y^2} = \lim_{x\to 0}\frac{x^2}{x^2} = 1.$$

Since these two limits are not same, therefore limit does not exist.

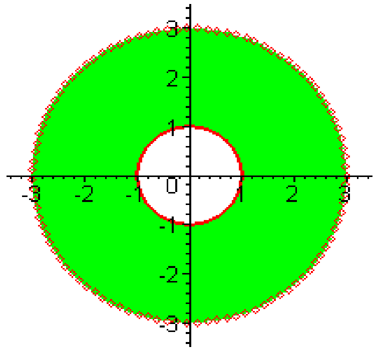
Q.2: Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the function $F(x,y,z) = x+y^2+z^3-2xy+3xz-5yz+6$.

Sol: $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{1-2y+3z}{3z^2+3x-5y}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2y-2x-5z}{3z^2+3x-5y}$.

Q.3: Find and sketch the domain of the function $f(x,y) = \sqrt{x^2+y^2-1} + \ln\left(9-x^2-y^2\right)$. Write the domain in words.

Sol: $x^2+y^2-1\geq 0 \Rightarrow x^2+y^2\geq 1$ and $9-x^2-y^2>0 \Rightarrow x^2+y^2<9$.

The domain is inside the open circle of radius 3 and outside the circle of radius 1.



Q.4: Show that $u(x,t) = \sin(x-at)+\cos(x+at)$ is a solution of the wave equation $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

Sol: $\dfrac{\partial u}{\partial x} = \cos(x-at)-\sin(x+at)$ and $\dfrac{\partial^2 u}{\partial x^2} = -\left(\sin(x-at)+\cos(x+at)\right)$

$\dfrac{\partial u}{\partial t} = -a\cos(x-at)-a\sin(x+at)$ and $\dfrac{\partial^2 u}{\partial t^2} = -a^2\left(\sin(x-at)+\cos(x+at)\right) = a^2\dfrac{\partial^2 u}{\partial x^2}$.

Q.5: Show that $f(x,y) = xe^{xy}$ is differentiable at the point $P(1,0)$ and find the linearization $L(x,y)$ of $f(x,y)$ at the point $P(1,0)$. Use $L(x,y)$ to approximate $f(1.1,-0.1)$.

Sol: $\dfrac{\partial f}{\partial x} = e^{xy}+xye^{xy}$ and $\dfrac{\partial f}{\partial y} = x^2e^{xy}$

$\dfrac{\partial f(1,0)}{\partial x} = 1$ and $\dfrac{\partial f(1,0)}{\partial y} = 1$. Since both derivatives exist at $P(1,0)$, therefore f is differentiable at $P(1,0)$.

$L(x,y) = f(a,b) + \dfrac{\partial f(a,b)}{\partial x}(x-a) + \dfrac{\partial f(a,b)}{\partial x}(y-b) = 1+1(x-1)+1(y-0) = x+y$

$L(1.1,-0.1) = 1.1-0.1 = 1.$

Q.6: If $z = f(x,y) = x^2+3xy-y^2$, find the differential dz. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compute the value of dz.

Sol: $dz = f_x dx + f_y dy = (2x+3y)\,dx + (3x-2y)\,dy.$

With $x=2,\ y=3$ and $dx=\Delta x = 0.05,\ dy = \Delta y = -0.04,$

$dz = (4+9)(0.05)+(0)(-0.04) = 0.65.$