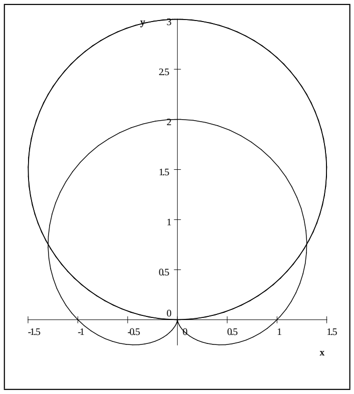
Solution Math 201 Class Test 1 (B)

Q.1: Find area of the region that lies inside the cardiod $r = 1 + \sin\theta$ and outside the circle $r = 3\sin\theta$.

Sol: The two graphs intersect at $3\sin\theta = 1 + \sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$



$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+\sin\theta)^2\, d\theta - \int_{0}^{\frac{\pi}{6}} (3\sin\theta)^2\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(1 + 2\sin\theta + \sin^2\theta\right) d\theta - 9\int_{0}^{\frac{\pi}{6}} \sin^2\theta d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta - 9\int_{0}^{\frac{\pi}{6}} \frac{1-\cos 2\theta}{2} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2}\right) d\theta - \frac{9}{2}\int_{0}^{\frac{\pi}{6}} (1-\cos 2\theta)\, d\theta$$

$$= \left(\frac{3}{2}\theta - 2\cos\theta - \frac{\sin 2\theta}{4}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{6}} - \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_{0}^{\frac{\pi}{6}}$$

$$= \left(\frac{\pi}{4} - \sqrt{3} - \frac{\sqrt{3}}{8} + \frac{3\pi}{4}\right) - \frac{9}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{1}{4}\pi.$$

Q.2: Find equation of the sphere with radius 6 and center at $(3, 5, 4)$. Also find its intersection with the $xz-$plane.

Sol: Equation of the sphere is $(x-3)^2 + (y-5)^2 + (z-4)^2 = 6^2$

This equation intersect the $xz-$plane at $y = 0 \Rightarrow (x-3)^2 + (0-5)^2 + (z-4)^2 = 6^2$

$(x-3)^2 + (z-4)^2 = 36 - 25 = 11$ which is a circle of radius $\sqrt{11}$ and center at $(3, 4)$.

Q.3: If a vector $\mathbf{u}$ had magnitude $|\mathbf{u}| = 5$ and makes angle $\frac{\pi}{3}$ with positive $x-$axis and another vector $\mathbf{v}$ has magnitude $|\mathbf{v}| = 3$ and makes angle $\frac{\pi}{6}$ with positive x-axis. Find a vector of magnitude 5 in the direction of $\mathbf{u} + \mathbf{v}$.

Sol: For vector $\mathbf{u}$, $x = |\mathbf{u}|\cos\theta = 5\cos\frac{\pi}{3} = \frac{5}{2}$ and $y = |\mathbf{u}|\sin\theta = 5\sin\frac{\pi}{3} = \frac{5\sqrt{3}}{2}$

So the vector $\mathbf{u}$ is $\mathbf{u} = \left\langle \frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$

For the vector **v**, $x = |\mathbf{v}|\cos\theta = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$ and $y = |\mathbf{v}|\sin\theta = 3\sin\frac{\pi}{6} = \frac{3}{2}$

So the vector **v** is $\mathbf{v} = \left\langle \frac{3\sqrt{3}}{2}, \frac{3}{2} \right\rangle$.

$$\mathbf{u}+\mathbf{v} = \left\langle \frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle + \left\langle \frac{3\sqrt{3}}{2}, \frac{3}{2} \right\rangle = \left\langle \frac{5+3\sqrt{3}}{2}, \frac{5\sqrt{3}+3}{2} \right\rangle$$

and $|\mathbf{u}+\mathbf{v}| = \sqrt{\left(\frac{5+3\sqrt{3}}{2}\right)^2 + \left(\frac{5\sqrt{3}+3}{2}\right)^2} = \sqrt{15\sqrt{3}+34}$

The required vector is $5\frac{\mathbf{u}+\mathbf{v}}{|\mathbf{u}+\mathbf{v}|} = \frac{5}{\sqrt{15\sqrt{3}+34}}\left\langle \frac{5+3\sqrt{3}}{2}, \frac{5\sqrt{3}+3}{2} \right\rangle$.

Q.4: If $\mathbf{u} = \langle \mathbf{3}, 0, 2\rangle$. Find a vector **v** such that $\text{comp}_{\mathbf{u}}\mathbf{v} = \mathbf{2}$.

Sol: Let $v = \langle a, b, c\rangle$ then $\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{u\cdot v}{|u|} = \frac{3a+2c}{\sqrt{13}} = 2$

$\Rightarrow 3a + 2c = 2\sqrt{13}$ or $c = \frac{2\sqrt{13}-3a}{2} = \sqrt{13} - \frac{3a}{2}$

The required vector is $v = \langle a, b, \sqrt{13} - \frac{3a}{2}\rangle$, where a and b are arbitrary.

Q.5: If $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. Show that the vector equation $(\mathbf{r}-\mathbf{a})\cdot(r-b) = 0$ represents a sphere. Find radius and center of that sphere.

Sol: $(\mathbf{r}-\mathbf{a})\cdot(r-b) = \langle x-a_1, y-a_2, z-a_3\rangle \cdot \langle x-b_1, y-b_2, z-b_3\rangle$

$= (x-a_1)(x-b_1) + (y-a_2)(y-b_2) + (z-a_3)(z-b_3)$

$= a_1b_1 - xb_1 - ya_2 - yb_2 - za_3 - zb_3 - xa_1 + a_2b_2 + a_3b_3 + x^2 + y^2 + z^2 = 0$

$x^2 + y^2 + z^2 - (a_1+b_1)x - (a_2+b_2)y - (a_3+b_3)z + a_1b_1 + a_2b_2 + a_3b_3 = 0$

$x^2 + y^2 + z^2 - (a_1+b_1)x - (a_2+b_2)y - (a_3+b_3)z = -a_1b_1 - a_2b_2 - a_3b_3$

$$x^2 - (a_1+b_1)x + \frac{(a_1+b_1)^2}{4} + y^2 - (a_2+b_2)y + \frac{(a_2+b_2)^2}{4} + z^2 - (a_3+b_3)z + \frac{(a_3+b_3)^2}{4}$$
$$= -a_1b_1 + \frac{(a_1+b_1)^2}{4} - a_2b_2 + \frac{(a_2+b_2)^2}{4} - a_3b_3 + \frac{(a_3+b_3)^2}{4}$$
$$\left(x - \left(\frac{a_1+b_1}{2}\right)\right)^2 + \left(y - \left(\frac{a_2+b_2}{2}\right)\right)^2 + \left(z - \left(\frac{a_3+b_3}{2}\right)\right)^2 = \frac{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}{4}$$

Which is a sphere with center at $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$

and radius $R = \sqrt{\frac{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}{4}}$