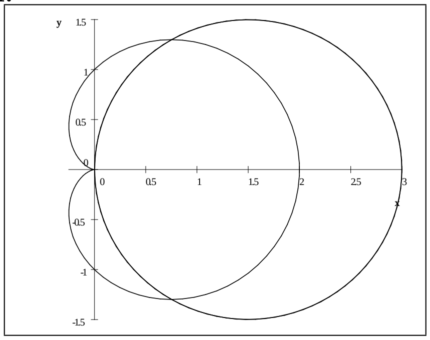
Solution Math 201 Class Test 1 (A)

Q.1: Find area of the region that lies inside the cardiod $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.

Sol:



The two graphs intersect at $3\cos\theta = 1 + \cos\theta \ \Rightarrow 2\cos\theta = 1 \ \Rightarrow \cos\theta = \frac{1}{2} \ \Rightarrow \theta = \frac{\pi}{3}$

$$A = \int_{\frac{\pi}{3}}^{\pi} (1+\cos\theta)^2\, d\theta - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (3\cos\theta)^2\, d\theta = \int_{\frac{\pi}{3}}^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9\cos^2\theta d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta - 9\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta - \frac{9}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$

$$= \left(\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right)\Bigg|_{\frac{\pi}{3}}^{\pi} - \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left(\frac{3\pi}{2} - \frac{\pi}{2} - 2\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) - \frac{9}{2}\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) = \frac{1}{4}\pi.$$

Q.2: Find equation of the sphere with radius 5 and center at $(3, 2, 4)$. Also find its intersection with the $xy-$plane.

Sol: Equation of the sphere is $(x-3)^2 + (y-2)^2 + (z-4)^2 = 5^2$

This equation intersect the $xy-$plane at $z = 0 \ \Rightarrow (x-3)^2 + (y-2)^2 + (0-4)^2 = 5^2$

$(x-3)^2 + (y-2)^2 = 25 - 16 = 9$ which is a circle of radius 3 and center at $(3, 2)$.

Q.3: If a vector $\mathbf{u}$ had magnitude $|\mathbf{u}| = 4$ and makes angle $\frac{\pi}{6}$ with positive $x-$axis and another vector $\mathbf{v}$ has magnitude $|\mathbf{v}| = 5$ and makes angle $\frac{\pi}{3}$ with positive x-axis. Find a vector of magnitude 6 in the direction of $\mathbf{u} + \mathbf{v}$.

Sol: For vector $\mathbf{u}$, $x = |\mathbf{u}|\cos\theta = 4\cos\frac{\pi}{6} = 2\sqrt{3}$ and $y = |\mathbf{u}|\sin\theta = 4\sin\frac{\pi}{6} = 2$

So the vector $\mathbf{u}$ is $\mathbf{u} = \langle 2\sqrt{3}, 2\rangle$.

For the vector $\mathbf{v}$, $x = |\mathbf{v}|\cos\theta = 5\cos\frac{\pi}{3} = \frac{5}{2}$ and $y = |\mathbf{v}|\sin\theta = 5\sin\frac{\pi}{3} = \frac{5}{2}\sqrt{3}$

So the vector $\mathbf{v}$ is $\mathbf{v} = \langle \frac{5}{2}, 5\frac{\sqrt{3}}{2} \rangle$.

$$\mathbf{u}+\mathbf{v} = \langle 2\sqrt{3}, 2\rangle + \left\langle \frac{5}{2}, 5\frac{\sqrt{3}}{2} \right\rangle = \left\langle \frac{4\sqrt{3}+5}{2}, \frac{4+5\sqrt{3}}{2} \right\rangle$$

and $|\mathbf{u}+\mathbf{v}| = \sqrt{\left(\frac{4\sqrt{3}+5}{2}\right)^2 + \left(\frac{4+5\sqrt{3}}{2}\right)^2} = \sqrt{\left(2\sqrt{3}+\frac{5}{2}\right)^2 + \left(\frac{5}{2}\sqrt{3}+2\right)^2} = \sqrt{20\sqrt{3}+41}$

The required vector is $6\frac{\mathbf{u}+\mathbf{v}}{|\mathbf{u}+\mathbf{v}|} = \frac{6}{\sqrt{20\sqrt{3}+41}} \left\langle \frac{4\sqrt{3}+5}{2}, \frac{4+5\sqrt{3}}{2} \right\rangle$

Q.4: If $\mathbf{u} = \langle 2, 1, 0\rangle$. Find a vector $\mathbf{v}$ such that $\text{comp}_{\mathbf{u}}\mathbf{v} = \mathbf{3}$.

Sol: Let $v = \langle a, b, c\rangle$ then $\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{u \cdot v}{|u|} = \frac{2a+b}{\sqrt{5}} = 3$

$\Rightarrow 2a + b = 3\sqrt{5}$ or $b = 3\sqrt{5} - 2a$

The required vector is $v = \langle a, 3\sqrt{5} - 2a, c\rangle$, where a and c are arbitrary.

Q.5: If $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. Show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (r - b) = 0$ represents a sphere. Find radius and center of that sphere.

Sol: $(\mathbf{r} - \mathbf{a}) \cdot (r - b) = \langle x - a_1, y - a_2, z - a_3\rangle \cdot \langle x - b_1, y - b_2, z - b_3\rangle$

$= (x - a_1)(x - b_1) + (y - a_2)(y - b_2) + (z - a_3)(z - b_3)$

$= a_1b_1 - xb_1 - ya_2 - yb_2 - za_3 - zb_3 - xa_1 + a_2b_2 + a_3b_3 + x^2 + y^2 + z^2 = 0$

$x^2 + y^2 + z^2 - (a_1 + b_1)x - (a_2 + b_2)y - (a_3 + b_3)z + a_1b_1 + a_2b_2 + a_3b_3 = 0$

$x^2 + y^2 + z^2 - (a_1 + b_1)x - (a_2 + b_2)y - (a_3 + b_3)z = -a_1b_1 - a_2b_2 - a_3b_3$

$$x^2 - (a_1 + b_1)x + \frac{(a_1+b_1)^2}{4} + y^2 - (a_2 + b_2)y + \frac{(a_2+b_2)^2}{4} + z^2 - (a_3 + b_3)z + \frac{(a_3+b_3)^2}{4}$$

$$= -a_1b_1 + \frac{(a_1+b_1)^2}{4} - a_2b_2 + \frac{(a_2+b_2)^2}{4} - a_3b_3 + \frac{(a_3+b_3)^2}{4}$$

$$\left(x - \left(\frac{a_1+b_1}{2}\right)\right)^2 + \left(y - \left(\frac{a_2+b_2}{2}\right)\right)^2 + \left(z - \left(\frac{a_3+b_3}{2}\right)\right)^2 = \frac{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}{4}$$

Which is a sphere with center at $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$

and radius $R = \sqrt{\frac{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}{4}}$