Solution Math 102 Quiz 5 (B)

Q.1: Determine whether the series converges or diverges $\sum_{n=0}^{\infty} \frac{(n+1)!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+3)}$

Sol: $a_n = \frac{(n+1)!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+3)}$ and $a_{n+1} = \frac{(n+2)(n+1)!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+3)(2n+5)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)(n+1)!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+3)(2n+5)} \cdot \frac{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+3)}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+2)}{(2n+5)} = \frac{1}{2} < 1, \text{ the aeries converges by RATIO test.}$$

Q.2: Determine the radius of convergence and interval of convergence for the seires $\sum_{n=1}^{\infty} \frac{x^n}{4^n n^3}$.

Sol: $a_n = \frac{x^n}{4^n n^3}$ and $a_{n+1} = \frac{x^{n+1}}{4^{n+1}(n+1)3} = \frac{x^n x}{4^n 4 (n+1)^3}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^n x}{4^n 4 (n+1)^3} \cdot \frac{4^n n^3}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{4(n+1)^3} \cdot \frac{n^3}{1}\right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 \left|\frac{x}{4}\right| = \left|\frac{x}{4}\right| < 1 \text{ for convergence}$$

$\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$ or $-4 < x < 4$ and $R = 4$

At $x = -4$, the series $\sum_{n=1}^{\infty} \frac{(-4)^n}{4^n n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges by Alternating Series Test.

At $x = 4$, the series $\sum_{n=1}^{\infty} \frac{(4)^n}{4^n n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges as a $p-$series with $p = 3$.

Therefore interval of convergence is $[-4, 4]$ and radius of convergence is $R = 4$.

Q.3: Find a power series representation for te function $f(x) = \ln(1+x)$.

Sol: $$f(x) = \ln(1+x) = \int \frac{1}{1+x} dx = \int \left[\sum_{n=1}^{\infty} (-1)^n x^n\right] dx = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

Q.4: Find sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+2}(2n+1)!}$

Sol: $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+2}(2n+1)!} = \sum_{n=0}^{\infty} \frac{1}{4} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \frac{1}{4} \sin\left(\frac{\pi}{4}\right) = \frac{1}{8}\sqrt{2}$$