Solution Math 102 Quiz 5 (A)

Q.1: Determine whether the series converges or diverges $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$

Sol: $$a_n = \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)} \text{ and } a_{n+1} = \frac{(n+1)\, n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)\,(3n+5)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\, n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)\,(3n+5)} \cdot \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{(3n+5)} = \frac{1}{3} < 1, \text{ the aeries converges by RATIO test.}$$

Q.2: Determine the radius of convergence and interval of convergence for the seires $\sum_{n=1}^{\infty} \frac{x^n}{3^n n^5}$.

Sol: $$a_n = \frac{x^n}{3^n n^5} \text{ and } a_{n+1} = \frac{x^{n+1}}{3^{n+1}(n+1)^5} = \frac{x^n x}{3^n 3 (n+1)^5}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^n x}{3^n 3 (n+1)^5} \cdot \frac{3^n n^5}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3(n+1)^5} \cdot \frac{n^5}{1} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^5 \left| \frac{x}{3} \right| = \left| \frac{x}{3} \right| < 1 \text{ for convergence}$$

$$\left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3 \text{ or } -3 < x < 3 \text{ and } R = 3$$

At $x = -3$, the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^5} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges by Alternating Series Test.

At $x = 3$, the series $\sum_{n=1}^{\infty} \frac{(3)^n}{3^n n^5} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ converges as a $p-$series with $p = 5$.

Therefore interval of convergence is $[-3, 3]$ and radius of convergence is $R = 3$.

Q.3: Find a power series representation for te function $f(x) = \tan^{-1}(x)$.

Sol: $$(x) = \tan^{-1}(x) = \int \frac{1}{1+x^2} dx = \int \left[\sum_{n=1}^{\infty} (-1)^n \left(x^2\right)^n \right] dx = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

Q.4: Find sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n+1}(2n)!}$

Sol: $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n+1}(2n)!} = \sum_{n=0}^{\infty} \frac{1}{3} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = \frac{1}{3} \cos\left(\frac{\pi}{3}\right) = \frac{1}{6}.$$