Solution Math 102 Quiz 4 (B)

Q.1: Determine whether the series converges or diverges $\sum_{n=1}^{\infty} \frac{3n^2+2n}{3^n (2n+3)^2}$.

Sol: Here $a_n = \frac{3n^2+2n}{3^n (2n+3)^2}$, let $b_n = \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^2+2n}{3^n (2n+3)^2} \frac{3^n}{1} = \lim_{n\to\infty} \frac{3n^2+2n}{(2n+3)^2} = \frac{3}{4} > 0$.

Since $\sum_{n=1}^{\infty} b_n$ converges (GS with $r = \frac{1}{3}$),

therefore $\sum_{n=1}^{\infty} a_n$ also converges by Limit Comparison Test.

Q.2: Determine whether the series converges or diverges $\sum_{n=1}^{\infty} \frac{(-1)\, n}{\ln n}$.

Sol: $a_n = \frac{(-1)\, n}{\ln n} = (-1)^{n-1} b_n$, where $b_n = \frac{n}{\ln n}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$. So the series Diverges.

Also if $f(x) = \frac{x}{\ln x}$, then $f'(x) = \frac{\ln x - 1}{x^2} > 0$ for all $x > e$.

$f(x)$ increasing $\Rightarrow b_n = \frac{n}{\ln n}$ is also increasing.

Therefore the series Diverges by Alternating Series Test.

Q.3: Determine whether the series converges or diverges $\sum_{n=1}^{\infty} \frac{n!}{n^{n+1}}$.

Sol: $a_n = \frac{n!}{n^{n+1}}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+2}} = \frac{(n+1)\, n!}{(n+1)^{n+1} (n+1)} = \frac{n!}{n^{n+1} \left(1+\frac{1}{n}\right)^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} . \frac{n^{n+1}}{n!} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)} = \frac{1}{e\,(1)} = \frac{1}{e} < 1$$

Therefore the series converges by the Ratio Test.