Solution Math 102 Class Test 3 (B)

Q.1: Evaluate the integral $\int \frac{2}{(x^2+8x+17)^2}dx$

Sol: $\int \frac{2}{(x^2+8x+17)^2}dx = \int \frac{2}{\left((x+4)^2+1\right)^2}dx = \int \frac{2\sec^2\theta}{(\sec^2\theta)^2}d\theta$ By using $x+4=\tan\theta$

$$= 2\int \cos^2\theta d\theta = \int (1+\cos 2\theta)\,d\theta = \left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= (\theta + \sin\theta\cos\theta) + C = \left(\tan^{-1}(x+4) + \frac{x+4}{(x+4)^2+1}\right) + C$$

$$= \left(\tan^{-1}(x+4) + \frac{x+4}{x^2+8x+17}\right) + C$$

Q.2: Evaluate the integral $\int \frac{x^2-2x+2}{x^3+x}dx$

Sol: $\frac{x^2-2x+2}{x^3+x} = \frac{x^2-2x+2}{x(x^2+1)} = \frac{2}{x} + \frac{-x-2}{x^2+3}$ Using partial fractions technique

$$\int \frac{x^2-2x+2}{x^3+x}dx = \int \left(\frac{2}{x} + \frac{-x-2}{x^2+1}\right)dx = \int \left(\frac{2}{x} - \frac{x}{x^2+1} - \frac{2}{x^2+3}\right)dx$$

$$= 2\ln|x| - \frac{1}{2}\ln\left(x^2+1\right) - 2\tan^{-1}x + C$$

Q.3: Evaluate the integral $\int \frac{3}{x\sqrt{x+9}}dx$. (Hint: Put $u=\sqrt{x+9}$)

Sol: Put $u=\sqrt{x+9}$ then $u^2=x+9$, $x=u^2-9$ and $2udu=dx$

$$\int \frac{3}{x\sqrt{x+9}}dx = 3\int \frac{2u}{(u^2-9)\,u}du = 6\int \frac{1}{(u^2-9)}du$$

$$= 6\int \frac{1}{(u-3)(u+3)}du = \int \left(\frac{1}{(u-3)} - \frac{1}{u+3}\right)du$$

$$= \ln|u-3| - \ln|u+3| + C = \ln\left|\frac{u-3}{u+3}\right| + C = \ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right| + C$$

Q.4: Evaluate the integral $\int\limits_1^{\infty} \frac{3\ln x}{x^3} dx$

Sol: $\int \frac{3\ln x}{x^3} dx = 3\left(\frac{-1}{2x^2}\ln x - \int \frac{-1}{2x^2}\frac{1}{x} dx\right) = 3\left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)$

$$\int\limits_1^{\infty} \frac{3\ln x}{x^3} dx = \lim_{t\to\infty} \int\limits_1^{t} \frac{3\ln x}{x^3} dx = \lim_{t\to\infty} 3\left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)\Bigg|_1^t = \lim_{t\to\infty} 3\left(-\frac{\ln t}{2t^2} - \frac{1}{4t^2} + \frac{1}{3}\right) = \frac{3}{4}$$

Since the integral converges, therefore the series also converges.

Q.5: Evaluate the integral $\int\limits_1^{4} \frac{1}{(x-3)^2} dx$

Sol: The integrand $\frac{1}{(x-3)^2}$ has a discontinuity at $x = 3$ in the interval $[1, 4]$

$$\int\limits_1^{3} \frac{1}{(x-3)^2} dx + \int\limits_3^{4} \frac{1}{(x-3)^2} dx = \lim_{t\to 3^-} \int\limits_1^{t} \frac{1}{(x-3)^2} dx + \lim_{t\to 3^+} \int\limits_t^{4} \frac{1}{(x-3)^2} dx$$

$$= \lim_{t\to 3^-} \left(\frac{-1}{x-3}\right)\Bigg|_1^t + \lim_{t\to 3^+} \left(\frac{-1}{x-3}\right)\Bigg|_t^4$$

$$= \lim_{t\to 3^-} \left(\frac{-1}{t-3} + \frac{1}{1-3}\right) + \lim_{t\to 3^+} \left(\frac{-1}{4-3} + \frac{1}{t-3}\right) = \infty + \infty$$

The integral diverges.

Q.6: Determine whether the sequence converges or diverges $\left\{\frac{2^{2n+1}}{3^{n+2}}\right\}_{n=1}^{\infty}$.

Sol: $a_n = \frac{2^{2n+1}}{3^{n+2}} = \frac{2^{2n}}{3^n}\frac{2}{3^2} = \frac{2}{9}\left(\frac{2^2}{3}\right)^n = \frac{2}{9}\left(\frac{4}{3}\right)^n$

Since $\frac{4}{3} > 1$, therefore $\lim_{n\to\infty} a_n = \frac{2}{9}\lim_{n\to\infty}\left(\frac{4}{3}\right)^n \to \infty$.

So the sequence diverges.

Q.7: Determine whether the series converges or diverges $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

Sol: $$a_n = \frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$S_n = a_2 + a_3 + \cdots + a_n = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \cdots$$

$$+ \frac{1}{n-3} - \frac{1}{n-1} + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1}$$

$$= 1 + \frac{1}{3} - \frac{1}{n} - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \frac{4}{3} \Rightarrow \text{the series converges and } \sum_{n=1}^{\infty} \frac{1}{n^2-1} = \frac{4}{3}.$$

Q.8: Use series to express the number as ratio $3.\bar{2}\bar{5} = 3.25252525\cdots$

Sol: $$3.\bar{2}\bar{5} = 3.25252525\cdots = 3 + 0.25 + 0.0025 + 0.000025 + \cdots$$

$$= 3 + \frac{25}{10^2} + \frac{25}{10^4} + \frac{25}{10^6} + \cdots$$

$$= 3 + \frac{25}{100}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right)$$

$$= 3 + \frac{25}{100}\left(\frac{1}{1 - \frac{1}{10^2}}\right) = 3 + \frac{25}{100}\left(\frac{100}{99}\right) = 3 + \frac{25}{99} = \frac{322}{99}$$