Solution Math 102 Class Test 1 (A)

Q.1: Evaluate the integral $\int_0^1 x^{\frac{3}{2}}\sqrt{2+x^{\frac{5}{2}}}\,dx$

Sol: Put $u = 2 + x^{\frac{5}{2}}$, then $du = \frac{5}{2}x^{\frac{3}{2}}dx$. When $x = 0,\ u = 2$ and when $x = 1,\ u = 3$.

So $$\int_0^1 x^{\frac{3}{2}}\sqrt{2+x^{\frac{5}{2}}}\,dx = \frac{2}{5}\int_2^3 \sqrt{u}du = \frac{2}{5}\frac{2}{3}\,u^{\frac{3}{2}}\Big|_2^3 = \frac{4}{15}\left(3^{\frac{3}{2}} - 2^{\frac{3}{2}}\right).$$

Q.2: Evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\,dx$

Sol: $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\,dx = \int_{-\frac{\pi}{2}}^{0} -\sin x\ dx + \int_0^{\frac{\pi}{2}} \sin x\ dx = \cos x|_{-\frac{\pi}{2}}^{0} + -\cos x|_0^{\frac{\pi}{2}} = (1-0) + (0+1) = 2.$$

Q.3: Evaluate the integral $\int_0^{\pi} \frac{\sin^2 x}{1+\cos x}dx$

Sol: $$\int_0^{\pi} \frac{\sin^2 x}{1+\cos x}dx = \int_0^{\pi} \frac{1-\cos^2 x}{1+\cos x}dx = \int_0^{\pi} \frac{(1-\cos x)(1+\cos x)}{1+\cos x}dx$$

$$= \int_0^{\pi} (1-\cos x)\,dx = (x - \sin x)|_0^{\pi} = \pi$$

Q.4: Evaluate the integral $\int \frac{\sec x\ \tan x}{1+\sec^2 x}dx$

Sol: Put $u = \sec x$, then $du = \sec x \tan x dx$

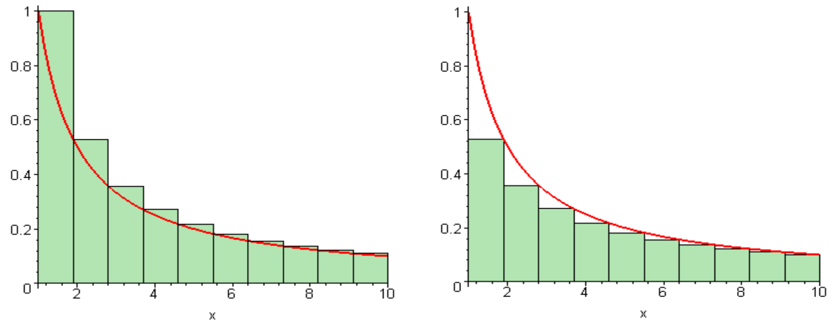
$$\int \frac{\sec x\ \tan x}{1+\sec^2 x}dx = \int \frac{1}{1+u^2}du = \tan^{-1} u + C = \tan^{-1}(\sec x) + C$$

Q.5: By comparing areas, show that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$. (Show all your work)

Sol: $\ln n = \int_1^n \frac{1}{t}dt$

The area under the graph of $f(t) = \frac{1}{t}$ from $t = 1$ to $t = n$ is $\int_1^n \frac{1}{t}dt$.

$\int_1^n \frac{1}{t}dt$ can be approximated by $n-1$ rectangles $L_{n-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$ which is an over estimate

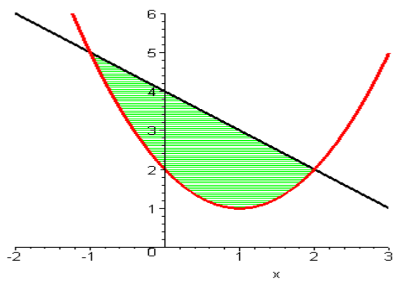


and also by $R_{n-1} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ which is an under estimate.of

$$\text{So } \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

Q.6: Find area of the region bounded by the $y = x^2 - 2x + 2$ and $y = -x + 4$.

Sol:

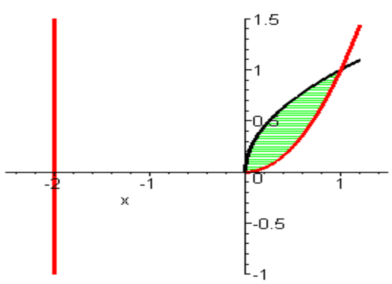


$$x^2 - 2x + 2 = -x + 4 \;\Rightarrow x^2 - x - 2 = 0 \;\Rightarrow (x+1)(x-2) = 0 \;\Rightarrow x = -1, 2.$$

$$A = \int_{-1}^{2} \left(-x + 4 - x^2 + 2x - 2\right) dx = \int_{-1}^{2} \left(x + 2 - x^2\right) dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Big|_{-1}^{2} = \frac{9}{2}$$

Q.7: Find volume of solid obtained by rotating the region bounded by $y = x^2$, and $x = y^2$ about the line $x = -2$.

Sol:



$$A(y) = \pi\left[\left(\sqrt{y} + 2\right)^2 - \left(y^2 + 2\right)^2\right] = \pi\left[y + 4\sqrt{y} + 4 - y^4 - 4y^2 - 4\right] = \pi\left[y + 4\sqrt{y} - y^4 - 4y^2\right]$$

$$V = \pi \int_0^1 \left(y + 4\sqrt{y} - y^4 - 4y^2\right) dy = \frac{49}{30}\pi.$$