Solution Math 002-01 Class Test Term: 081

Q.1: Verify the identity $\dfrac{\sin^3 x - \cos^3 x}{\sin x + \cos x} = \dfrac{\csc^2 x - \cot x - 2\cos^2 x}{1 - \cot^2 x}$

Sol:
$$\begin{aligned}
\frac{\sin^3 x - \cos^3 x}{\sin x + \cos x} &= \frac{(\sin x - \cos x)\left(\sin^2 x + \sin x \cos x + \cos^2 x\right)}{\sin x + \cos x} \\
&= \frac{(\sin x - \cos x)(1 + \sin x \cos x)}{\sin x + \cos x} \frac{(\sin x - \cos x)}{(\sin x + \cos x)} \\
&= \frac{(\sin x - \cos x)^2 (1 + \sin x \cos x)}{\sin^2 x - \cos^2 x} \\
&= \frac{\left(\sin^2 x - 2\sin x \cos x + \cos^2 x\right)(1 + \sin x \cos x)}{\sin^2 x - \cos^2 x} \\
&= \frac{(1 - 2\sin x \cos x)(1 + \sin x \cos x)}{\sin^2 x - \cos^2 x} = \frac{1 - \sin x \cos x - 2\sin^2 x \cos^2 x}{\sin^2 x - \cos^2 x} \\
&= \frac{\cot^2 x - \cot x - 2\cos^2 x}{1 - \cot^2 x}
\end{aligned}$$

Q.2: Evaluate exactly $\dfrac{\tan 35^\circ - \cot 265^\circ}{\sec^2 35^\circ - \tan^2 35^\circ + \tan 35^\circ \cot 265^\circ}$

Sol:
$$\begin{aligned}
&\frac{\tan 35^\circ - \cot 265^\circ}{\sec^2 35^\circ - \tan^2 35^\circ + \tan 35^\circ \cot 265^\circ} \\
&= \frac{\tan 35^\circ - \cot 85^\circ}{1 + \tan 35^\circ \cot 85^\circ} \qquad (\sec^2 x - \tan^2 x = 1 \text{ and reference angle of } 265^\circ \text{ is } 85^\circ \text{ in quadrant III}) \\
&= \frac{\tan 35^\circ - \tan 5^\circ}{1 + \tan 35^\circ \tan 5^\circ} \qquad (\text{since cofunction of } \cot x = \tan(90^\circ - x)) \\
&= \tan(35^\circ - 5^\circ) = \tan(30^\circ) = \frac{1}{3}\sqrt{3}.
\end{aligned}$$

Q.3: If $\cot x = -\dfrac{3}{4}$ and $\dfrac{3\pi}{2} \le x \le 2\pi$, find the value of $\csc\left(\dfrac{x}{2}\right)$.

Sol: $\sec^2 x = 1 + \tan^2 x = 1 + \dfrac{16}{9} = \dfrac{25}{9}$

$\sec x = \sqrt{1 + \tan^2 x} = \dfrac{5}{3}$ and $\cos x = \dfrac{3}{5}$

If $\dfrac{3\pi}{2} \le x \le 2\pi$ then $\dfrac{3\pi}{4} \le \dfrac{x}{2} \le \pi$. So $\dfrac{x}{2}$ lies in quadrant II.

$\sin\left(\dfrac{x}{2}\right) = \sqrt{\dfrac{1 - \cos x}{2}} = \sqrt{\dfrac{1 - \frac{3}{5}}{2}} = \sqrt{\dfrac{1}{5}} = \dfrac{1}{\sqrt{5}}$. So $\csc\left(\dfrac{x}{2}\right) = \sqrt{5}$.

Q.4: If $-\sin x + \sqrt{3}\cos x = k \sin(x + \theta)$, find value of k and θ.

Sol: Let $-1 = k\cos\theta$ and $\sqrt{3} = k\sin\theta$.Then

$$-\sin x + \sqrt{3}\cos x = k \sin x \cos\theta + k \cos x \sin\theta = k\sin(x + \theta)$$

where $k = \sqrt{1+3} = 2$ and $\tan\theta = -\sqrt{3} \Rightarrow \theta = \dfrac{2\pi}{3}$.

Q.5: Find the domain and range of $y = 2\cos^{-1}(2x + 3) + 2$.

Sol: Domain: $-1 \le 2x + 3 \le 1 \;\Rightarrow -4 \le 2x \le -2 \;\Rightarrow -2 \le x \le -1$.

Range for $y = \cos^{-1}(X)$ is $0 \le y \le \pi$

Range for $y = 2\cos^{-1}(X)$ is $0 \le y \le 2\pi$

Therefore range for $y = 2\cos^{-1}(X) + 2$ is $2 \le y \le 2\pi + 2$.

Q.6: Solve the equation $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$.

Sol: Let $\alpha = \sin^{-1} x$ then $\sin \alpha = x$ and $\cos \alpha = \sqrt{1 - x^2}$.

So $\alpha + \cos^{-1} x = \dfrac{\pi}{2} \;\Rightarrow \cos^{-1} x = \dfrac{\pi}{2} - \alpha \;\Rightarrow x = \cos\left(\dfrac{\pi}{2} - \alpha\right)$

$x = \cos\left(\dfrac{\pi}{2} - \alpha\right) = \cos\left(\dfrac{\pi}{2}\right)\sin \alpha + \sin\left(\dfrac{\pi}{2}\right)\sin \alpha = x$

The equation is true for all values of x. Therefore it is an identity.

OR:

Let $\alpha = \sin^{-1} x$ and $\beta = \cos^{-1} x$. Then $\sin \alpha = x$ and $\cos \beta = x$

Using right triangles for α and β, we get $\cos \alpha = \sqrt{1 - x^2}$ and $\sin \beta = \sqrt{1 - x^2}$.

$\sin^{-1} x + \cos^{-1} x = \alpha + \beta = \cos^{-1}\left[\cos(\alpha + \beta)\right] = \cos^{-1}\left[\cos \alpha \cos \beta - \sin \alpha \sin \beta\right]$

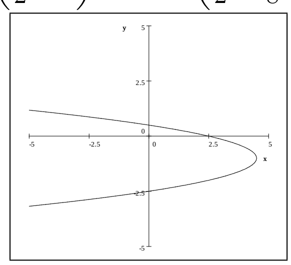
$= \cos^{-1}\left[x\sqrt{1 - x^2} - x\sqrt{1 - x^2}\right] = \cos^{-1}(0) = \dfrac{\pi}{2}$.

Q.7: Find the vertex, focus, and directrix of the parabola given by $2x + 4y^2 + 8y - 5 = 0$. Also sketch its graph.

Sol: Completing square $4\left(y^2 + 2y + 1\right) = -2x + 5 + 4$

$(y + 1)^2 = \dfrac{1}{4}(-2x + 9) = -\dfrac{1}{2}\left(x - \dfrac{9}{2}\right)$. $4p = -\dfrac{1}{2} \;\Rightarrow p = -\dfrac{1}{8}$

Vertex $\left(\dfrac{9}{2}, -1\right)$, Focus $\left(\dfrac{9}{2} - \dfrac{1}{8}, -1\right) = \left(\dfrac{35}{8}, -1\right)$, Directrix $x = \dfrac{9}{2} - \left(-\dfrac{1}{8}\right) = \dfrac{37}{8}$.

The graph is 

Q.8: If $\vec{u} = \langle 2, -3 \rangle$ and $\vec{v} = \langle -1, 4 \rangle$. Find the following:

(*a*) $\|2\vec{u} - 3\vec{v}\|$

(*b*) $\vec{u} \cdot \vec{v}$

(*c*) $\text{proj}_{\vec{v}}\vec{u}$.

Sol: (*a*) $2\vec{u} - 3\vec{v} = 2\langle 2, -3 \rangle - 3\langle -1, 4 \rangle = \langle 4, -6 \rangle + \langle 3, -12 \rangle = \langle 7, -18 \rangle$

$\|2\vec{u} - 3\vec{v}\| = \sqrt{(7)^2 + (18)^2} = \sqrt{373} = 19.313$

(*b*) $\vec{u} \cdot \vec{v} = \langle 2, -3 \rangle \cdot \langle -1, 4 \rangle = -2 - 12 = -14$.

(*c*) $\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} = \dfrac{-14}{\sqrt{1 + 16}} = \dfrac{-14}{\sqrt{17}}$.

Q.9: Find the equation in standard of the ellipse whose center is $(2, 4)$, major axis is parallel to the y-axis and of length 10 and ellipse is passing through $(3, 3)$. Also find eccentricity of the ellipse.

Sol: Center is $(2, 4)$, so $h = 2$ and $k = 4$. Since length of major axis is 10, therefore $2a = 10$ or $a = 5$.

Equation of ellipse is $\dfrac{(x - 2)^2}{b^2} + \dfrac{(y - 4)^2}{25} = 1$

Since graph of ellipse is passing through $(3, 3)$, therefore $\dfrac{(3 - 2)^2}{b^2} + \dfrac{(3 - 4)^2}{25} = 1$

$\Rightarrow \dfrac{1}{b^2} = 1 - \dfrac{1}{25} = \dfrac{24}{25} \;\Rightarrow b^2 = \dfrac{25}{24}$. Thus equation of ellipse is $\dfrac{(x - 2)^2}{\frac{25}{24}} + \dfrac{(y - 4)^2}{25} = 1$.

$c = \sqrt{a^2 - b^2} = \sqrt{25 - \dfrac{25}{24}} = \dfrac{5}{12}\sqrt{138}$ and essentricity is $e = \dfrac{c}{a} = \dfrac{\frac{5}{12}\sqrt{138}}{5} = \dfrac{\sqrt{138}}{12} = 0.97895$.