Solution Final Exam for Math 201 - 081

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NOTE: Correct choice for all questions is (A)

Q.1: (**Section 10.2–B**) The curve $C: \ x = t - \ln t, \ y = t + \ln t$ is concave down on:

(A) $(1, \infty)$

(B) $(0, 1)$

(C) $(0, \infty)$

(D) $(-\infty, 0) \cup (1, \infty)$

(E) $(-\infty, 1)$

Sol: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1 + \frac{1}{t}}{1 - \frac{1}{t}} = \dfrac{t+1}{t-1}$ and $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{(t-1)-(t+1)}{(t-1)^2\left(1 - \frac{1}{t}\right)} = \dfrac{-2t}{(t-1)^3}.$

The curve is concave down for all values of t for which $\dfrac{d^2y}{dx^2} < 0$,

$Interval$	$(-\infty, 0]$	$[0, 1)$	$(1, \infty)$
t	$-$	$+$	$+$
$t-1$	$-$	$-$	$+$
$\dfrac{d^2y}{dx^2}$	$-$	$+$	$-$

Since $x(t)$ and $y(t)$ are not defined for $(-\infty, 0]$, therefore $\dfrac{d^2y}{dx^2} < 0$ for $(1, \infty)$.

Q.2: (**Section 10.3–C**) Slope of the tangent line to the polar curve $r = 1 + \sin\theta$ at $\theta = \dfrac{\pi}{4}$ is:

(A) $-\dfrac{\sqrt{2}+2}{\sqrt{2}}$

(B) $-\dfrac{1}{\sqrt{2}}$

(C) $1 + \dfrac{\sqrt{2}}{2}$

(D) $\dfrac{\sqrt{2}-2}{\sqrt{2}}$

(E) $\dfrac{\sqrt{2}}{2-\sqrt{2}}$

Sol: $x = r\cos(\theta) = \cos(\theta) + \sin(\theta)\cos(\theta)$ and $y = r\sin(\theta) = \sin(\theta) + \sin^2(\theta)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos(\theta) + 2\sin(\theta)\cos(\theta)}{-\sin(\theta) + \cos^2(\theta) - \sin^2(\theta)} = \frac{\cos(\theta) + \sin(2\theta)}{-\sin(\theta) + \cos(2\theta)}$$

and $\left.\dfrac{dy}{dx}\right|_{\theta = \frac{\pi}{4}} = \dfrac{\frac{\sqrt{2}}{2} + 1}{-\frac{\sqrt{2}}{2}} = -\dfrac{\sqrt{2}+2}{\sqrt{2}}$

Q.3: (**Section 10.4–A**) The area of the region that lies inside both curves $r = 4\cos\theta$ and $r = 4\sin\theta$ is equal to:

(A) $2\pi - 4$

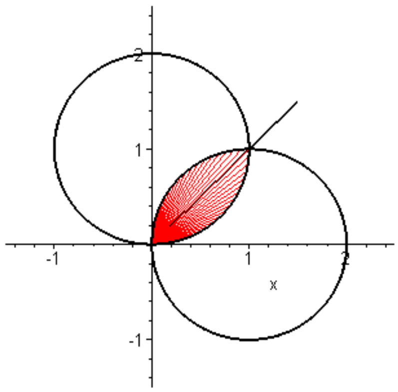
(B) $2\pi + 4$

(C) 4π

(D) $\pi + 2$

(E) $\pi - 2$

Sol: The are that lies in both the curves is the shaded area



$$A = 2.\frac{1}{2}\int_0^{\frac{\pi}{4}} 16\sin^2(\theta)\,d\theta = 16\int_0^{\frac{\pi}{4}} \frac{1-\cos(2\theta)}{2}d\theta = 8\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_0^{\frac{\pi}{4}} = 8\left(\frac{\pi}{4} - \frac{1}{2}\right) = 2\pi - 4.$$

OR

$$A = 2.\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 16\cos^2(\theta)\,d\theta = 16\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}d\theta = 8\left(\theta + \frac{\sin(2\theta)}{2}\right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 8\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = 2\pi - 4.$$

Q.4: (**Section 12.3–C**) Vector projection of $\vec{u} = \langle 1, 2, 3\rangle$ onto $\vec{v} = \langle 1, 4, 0\rangle$ is:

(A) $\left\langle \frac{9}{17}, \frac{36}{17}, 0 \right\rangle$

(B) $\left\langle \frac{9}{\sqrt{17}}, \frac{36}{\sqrt{17}}, 0 \right\rangle$

(C) $\left\langle \frac{9}{14}, \frac{36}{14}, 0 \right\rangle$

(D) $\left\langle \frac{9}{17}, \frac{18}{17}, \frac{27}{17} \right\rangle$

(E) $\left\langle \frac{9}{14}, \frac{18}{14}, \frac{27}{14} \right\rangle$

Sol: $Proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{9}{17}\langle 1, 4, 0\rangle = \langle \frac{9}{17}, \frac{36}{17}, 0\rangle.$

Q.5: (**Section 12.4–C**) The value of k for which the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$ and $\vec{c} = \langle k, 0, 1 \rangle$ are coplanar is:

(A) $k = 1$

(B) $k = -1$

(C) $k = -\dfrac{9}{23}$

(D) $k = \dfrac{7}{23}$

(E) $k = \dfrac{1}{9}$

Sol: The three vectors are coplanar if $\vec{a} \cdot \vec{b} \times \vec{c} = 0$,

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ k & 0 & 1 \end{vmatrix} = -1 - 4(2 - 4k) - 7(k) = 9k - 9 \overset{\text{Set}}{=} 0 \Rightarrow k = 1.$$

Q.6: (**Section 12.5–B**) Suppose that L_1 is the line passing through the points $(1, 0, 3)$ and $(0, 0, 4)$ and L_2 is the line passing through $(1, 0, 1)$ and has direction vector $\vec{v} = \langle 3, -1, 1 \rangle$. Then:

(A) L_1 and L_2 are skew lines

(B) L_1 and L_2 are parallel lines

(C) L_1 and L_2 are perpendicular lines

(D) L_1 and L_2 intersect at $(4, -1, 2)$

(E) L_1 and L_2 are identical lines

Sol: Direction vector for L_1 is $\vec{v}_1 = \langle -1, 0, 1 \rangle$ and direction vector for L_2 is $\vec{v}_2 = \langle 3, -1, 1 \rangle$.
Parametric equations for L_1 are: $x = -t,\ y = 0,\ z = 4 + t$.
Parametric equations for L_2 are: $x = 1 + 3s,\ y = -s,\ z = 1 + s$.
From $y = 0$ and $y = -s$, we get $s = 0$. From $x = -t$ and $x = 1 + 3s$, we get $t = -1$.
For $t = -1,\ z = 4 + t = 5$ and for $s = 0,\ z = 1 + s = 1$. Since $5 \neq 1$, therefore the two lines do not intersect. Also $\vec{v}_1 \nparallel \vec{v}_2$. Thus L_1 and L_2 are skew lines.

Q.7: (**Section 12.5–A**) An equation of the plane (P_1) that passes through the line of intersection of the planes (P_2) $x - z = 1$ and (P_3) $y + 2z = 0$, and is perpendicular to the plane (P_4) $x + y - 2z = 1$ is:

(a) $x + y + z = 1$

(b) $x + 2y = 9$

(c) $x + y = 4$

(d) $3x - y + z = 1$

(e) $x - 2y + z = -5$

Sol: A point on the line of intersection of P_2 and P_3 is $(1, 0, 0)$ and a vector parallel to the required plane passing through the line of intersection of P_2 and P_3 is:

$$\vec{n}_1 = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = i - 2j + k.$$

Since required plane P_1 is perpendicular to the plane P_4, therefore normal vector $\vec{n}_2 = i + j - 2k$ of P_4 is parallel to P_1. So the normal vector of P_1 is:

$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3i + 3j + 3k.$ and equation of P_1 is $3(x-1) + 3(y-0) + 3(z-0) = 0$ or $x + y + z = 1$.

Q.8: (**Section 12.6–C**) The quadratic surface $x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 = 0$ represents:

(A) Circular cone with vertex at $(2, -1, 1)$ and axis parallel to $y - axis$.

(B) Ellipsoid with center at $(-2, -1, 1)$

(C) Elliptic cone with vertex at $(1, -1, 1)$ and axis parallel to $z - axis$.

(D) Circular cone with vertex at $(2, 1, 1)$ and axis parallel to $y - axis$.

(E) Circular paraboloid with vertex at $(-4, -2, -2)$ and axis parallel to $z - axis$.

Sol: $x^2 - 4x - y^2 - 2y + z^2 - 2z = -4$

$x^2 - 4x + 4 - y^2 - 2y - 1 + z^2 - 2z + 1 = -4 + 4 + 1 - 1$

$(x-2)^2 - (y+1)^2 + (z-1)^2 = 0$

$(x-2)^2 + (z-1)^2 = (y+1)^2$

Circular cone with vertex at $(2, -1, 1)$ and axis parallel to $y - axis$.

Q.9: (**Section 14.2–C**) Let $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 3 & (x,y) = (0,0) \end{cases}$

and $L = \lim\limits_{(x,y)\to(0,0)} f(x,y)$. Then:

(A) L does not exist

(B) $L = 3$

(C) $L = 0$ and $f(x,y)$ is not continuous at $(0,0)$.

(D) $L = 1$ and $f(x,y)$ is not continuous at $(0,0)$.

(E) $L = 3$ and $f(x,y)$ is continuous at $(0,0)$.

Sol: If limit L exist and is equal to 3 then $f(x,y)$ is continuous at 3.

Let $(x,y) \to (0,0)$ through $y - axis$, that is $x = 0$ and $y \to 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{0 - y^2}{0 + y^2} = -1$$

Now let $(x,y) \to (0,0)$ through $x - axis$, that is $y = 0$ and $x \to 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - 0}{x^2 + 0} = 1.$$

Since these limits are different, therefore limit $L = \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Q.10: (**Section 14.3–C**) If $u = e^{ax+by+cz}$, where $a^2+b^2+c^2=6$, Then $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$ is equal to:

(A) $6u$

(B) u

(C) $\dfrac{6}{u}$

(D) $6u^2$

(E) u^2

Sol: $\dfrac{\partial u}{\partial x} = ae^{ax+by+cz}$ and $\dfrac{\partial^2 u}{\partial x^2} = a^2e^{ax+by+cz} = a^2u$. Similarly, $\dfrac{\partial^2 u}{\partial y^2} = b^2u$ and $\dfrac{\partial^2 u}{\partial z^2} = c^2u$.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = a^2u+b^2u+c^2u = \left(a^2+b^2+c^2\right)u = 6u.$$

Q.11: (**Section 14.5–B**) Let $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$. Then $\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$ is equal to:

(A) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$

(B) $\left(\dfrac{\partial z}{\partial x}\right)^2 - \left(\dfrac{\partial z}{\partial y}\right)^2$

(C) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \dfrac{1}{x^2+y^2}\left(\dfrac{\partial z}{\partial y}\right)^2$

(D) $\left(\dfrac{\partial z}{\partial x}\right)^2 . \left(\dfrac{\partial z}{\partial y}\right)^2$

(E) $\left(x^2+y^2\right)\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$

Sol: $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = \cos\theta\dfrac{\partial z}{\partial x} + \sin\theta\dfrac{\partial z}{\partial y}$ and $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta} = -r\sin\theta\dfrac{\partial z}{\partial x} + r\cos\theta\dfrac{\partial z}{\partial y}$

$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}\right)^2 \\ &= \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 + 2\sin\theta\cos\theta\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) \\ &\quad + \sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2 - 2\sin\theta\cos\theta\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) \\ &= \left(\cos^2\theta+\sin^2\theta\right)\left(\frac{\partial z}{\partial x}\right)^2 + \left(\sin^2\theta+\cos^2\theta\right)\left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.\end{aligned}$$

Q.12: (**Section 14.4–B**) Using the linear approximation of $f(x,y) = \sqrt{x^2+y^2}$ at the point $(3,4)$, the value of $\sqrt{(2.9)^2+(4.1)^2}$ is approximately equal to:

(A) $\dfrac{251}{50}$

(B) $\dfrac{257}{50}$

(C) $\dfrac{1}{50}$

(D) $\dfrac{6}{5}$

(E) $\dfrac{3}{25}$

Sol: $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, where $a=3,\ b=4$.

$f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ and $f_x(3,4) = \dfrac{3}{5}$. Also $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$ and $f_x(3,4) = \dfrac{4}{5}$.

$L(x,y) = f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4) = 5 + \dfrac{3}{5}(x-3) + \dfrac{4}{5}(y-4) = \dfrac{3}{5}x + \dfrac{4}{5}y.$

$f(2.9,4.1) = \sqrt{(2.9)^2+(4.1)^2} \approx L(2.9,4.1) = \dfrac{3}{5}(2.9) + \dfrac{4}{5}(4.1) = \dfrac{3(29)}{50} + \dfrac{4(41)}{50} = \dfrac{251}{50}.$

Q.13: (**Section 14.6–B**) The directional derivative of $f(x,y) = x^2 + \sin(xy)$ at the point $(1,0)$ is equal to 1 in the direction of unit vectors:

(A) $\langle 0,1\rangle$ and $\left\langle \dfrac{4}{5}, -\dfrac{3}{5}\right\rangle$

(B) $\langle 1,0\rangle$ and $\left\langle \dfrac{4}{5}, -\dfrac{3}{5}\right\rangle$

(C) $\langle -1,0\rangle$ and $\left\langle -\dfrac{4}{5}, \dfrac{3}{5}\right\rangle$

(D) $\langle 1,0\rangle$ and $\left\langle -\dfrac{4}{5}, -\dfrac{3}{5}\right\rangle$

(E) $\langle 0,1\rangle$ and $\left\langle -\dfrac{4}{5}, -\dfrac{3}{5}\right\rangle$

Sol: Let the unit vector be $u = \langle a,b\rangle$. Then $a^2+b^2=1$.

$D_u f(x,y) = \nabla f(x,y)\cdot u = \langle 2x + y\cos(xy), x\cos(xy)\rangle \cdot \langle a,b\rangle$

$D_u f(1,0) = \langle 2,1\rangle . \langle a,b\rangle = 2a+b$

According to the given condition, directional derivative is equal to 1, so $2a+b=1$ or $b=1-2a$.

$a^2+b^2=1 \ \Rightarrow a^2+(1-2a)^2=1 \ \Rightarrow a^2+1-4a+4a^2=1 \ \Rightarrow 5a^2-4a=0 \ \Rightarrow a=0,\ or\ a=\dfrac{4}{5}.$

If $a=0$, then $b=1$, and if $a=\dfrac{4}{5}$, then $b=-\dfrac{3}{5}$. So the unit vectors are $\langle 0,1\rangle$ and $\left\langle \dfrac{4}{5}, -\dfrac{3}{5}\right\rangle$.

Q.14: (**Section 14.7–B**) The function $f(x,y) = x^4 + y^4 - 4xy + \sqrt{5}$ has:

(A) Local minimum at $(1,1), (-1,-1)$ and saddle point at $(0,0)$.

(B) Local minimum at $(1,1), (-1,-1),\ (1,-1), (-1,1)$ and saddle point at $(0,0)$.

(C) Local maximum at $(1,1),\ (-1,-1)$ and saddle point at $(0,0)$.

(D) Local minimum at $(-1,-1)$, local maximum at $(1,1)$ and saddle point at $(0,0)$.

(E) Local minimum at $(1,1)$, local maximum at $(-1,-1)$ and saddle point at $(0,0)$.

Sol: $f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$.
$f_x = 0$ and $f_y = 0 \Rightarrow x^3 = y$ and $y^3 = x$.
$x^9 = y^3 = x \Rightarrow x\left(x^8 - 1\right) = 0 \Rightarrow x = 0, -1, 1$.
If $x = 0$, then $y = 0$. If $x = -1$, then $y = -1$. If $x = 1$, then $y = 1$.
So the critical points are $(0,0)$, $(1,1)$, $(-1,-1)$.
Now $f_{xx} = 12x^2$, $f_{yy} = 12y^2$, and $f_{xy} = -4$.
$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$.
$D(0,0) = 0$, so $(0,0)$ is a saddle point.
$D(1,1) = (12)(12) - (-4)^2 = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, so $(1,1)$ is a local minimum.
$D(-1,-1) = (12)(12) - (-4)^2 = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$, so $(1,1)$ is a local minimum.

Q.15: (**Section 14.7–B**) Determine the nature of the critical points $(1,2)$, $(-2,3)$, and $(-1,-1)$ of the function $g(x,y)$ if

$$\begin{array}{ccc} g_{xx}(1,2) = 2 & g_{yy}(1,2) = 3 & g_{xy}(1,2) = 2 \\ g_{xx}(-2,3) = -4 & g_{yy}(-2,3) = 5 & g_{xy}(-2,3) = 4 \\ g_{xx}(-1,-1) = 3 & g_{yy}(-1,-1) = 4 & g_{xy}(-1,-1) = 3 \end{array}$$

(A) Local minimum at $(-1,-1)$, Local minimum at $(1,2)$, Saddle point at $(-2,3)$.

(B) Local maximum at $(1,2)$, Local minimum at $(-1,-1)$, Saddle point at $(-2,3)$.

(C) Local maximum at $(-2,3)$, $(-1,-1)$, Local minimum at $(1,2)$.

(D) Local minimum at $(1,2)$, $(-1,-1)$, Saddle point at $(-2,3)$.

(E) Local minimum at $(-1,-1)$, Local maximum at $(1,2)$, $(-2,3)$.

Sol: $D(1,2) = g_{xx}(1,2)\, g_{yy}(1,2) - \left(g_{xy}(1,2)\right)^2 = 6 - 4 = 2 > 0$ and $g_{xx}(1,2) = 2 > 0$. So $(1,2)$ is a local minimum.
$D(-2,3) = g_{xx}(-2,3)\, g_{yy}(-2,3) - \left(g_{xy}(-2,3)\right)^2 = -20 - 16 = -36 < 0$. So $(-2,3)$ is a saddle point.
$D(-1,-1) = g_{xx}(-1,-1)\, g_{yy}(-1,-1) - \left(g_{xy}(-1,-1)\right)^2 = 12 - 9 = 3 > 0$ and $g_{xx}(-1,-1) = 3 > 0$. So $(-1,-1)$ is a local minimum.

Q.16: (**Section 14.8–B**) The maximum value of $f(x,y,z) = x + 2y - 3z$ subject to the constraint $z = 4x^2 + y^2$ is equal to: (Hint: Use Lagrange Multipliers)

(A) $\frac{17}{48}$

(B) 0

(C) $\frac{7}{8}$

(D) 5

(E) -2

Sol: $f(x,y,z) = x + 2y - 3z$, let $g(x,y,z) = 4x^2 + y^2 - z = 0$.
Then $\nabla f = \lambda \nabla g \Rightarrow \langle 1, 2, -3\rangle = \lambda \langle 8x, 2y, -1\rangle$
$\Rightarrow 8\lambda x = 1,\ 2\lambda y = 2,\ -\lambda = -3$
$\Rightarrow \lambda = 3$, and $x = \frac{1}{24}$, $y = \frac{1}{3}$. Also $g(x,y,z) = 4x^2 + y^2 - z = 0 \Rightarrow z = 4\left(\frac{1}{24}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{17}{144}$.
So maximum value of function is $f\left(\frac{1}{24}, \frac{1}{3}, \frac{1}{144}\right) = \frac{1}{24} + \frac{2}{3} - \frac{51}{144} = \frac{17}{48}$.

Q.17: (**Section 15.2–C**) The volume of the solid that lies under the paraboloid $z = 2b^2x^2 + a^2y^2$ $(a, b > 0)$ and above the rectangle $[0, a] \times [0, b]$ is:

(A) $(ab)^3$

(B) $(a+b)^3$

(C) $a^2b + ab^2$

(D) $a^3 + b^3$

(E) 1

Sol:
$$\int_0^a \int_0^b \left(2b^2x^2 + a^2y^2\right) dydx = \int_0^a \left(2b^2x^2y + \frac{a^2y^3}{3}\right)\Bigg|_0^b dx = \int_0^a \left(2b^3x^2 + \frac{a^2b^3}{3}\right) dx$$
$$= \left(\frac{2b^3x^3}{3} + \frac{a^2b^3}{3}x\right)\Bigg|_0^a = \frac{2b^3a^3}{3} + \frac{a^3b^3}{3} = (ab)^3 .$$

Q.18: (**Section 15.2–C**) The volume of the solid bounded by the surface $z = x\sqrt{x^2 + y}$ and the planes $x = 0,\ x = 1,\ y = 0,\ y = 1$, and $z = 0$ is:

(A) $\frac{2}{15}\left(2^{\frac{5}{2}} - 2\right)$

(B) $\frac{2}{15}\left(2^{\frac{5}{2}} + 2\right)$

(C) $\frac{2}{15}\left(2^{\frac{5}{2}} - 1\right)$

(D) $\frac{3}{15}\left(2^{\frac{5}{2}} + 2\right)$

(E) $\frac{4}{15}\left(2^{\frac{5}{2}} - 1\right)$

Sol:
$$\int_0^1 \int_0^1 \left(x\sqrt{x^2 + y}\right) dxdy = \int_0^1 \left(\frac{1}{3}x\left(x^2 + y\right)^{\frac{3}{2}}\right)\Bigg|_0^1 dy = \int_0^1 \left(\frac{1}{3}(1+y)^{\frac{3}{2}}\right) dy = \left(\frac{1}{3}\frac{2}{5}(1+y)^{\frac{5}{2}}\right)\Bigg|_0^1 =$$
$$\frac{2}{15}\left(2^{\frac{5}{2}} - 2\right).$$

Q.19: (**Section 15.3–B**) The volume of the solid under the surface $z = 2x + y^2$ and above the region in $xy - plane$ bounded by $x = y^2$ and $x = y^3$ is:

(A) $\frac{19}{210}$

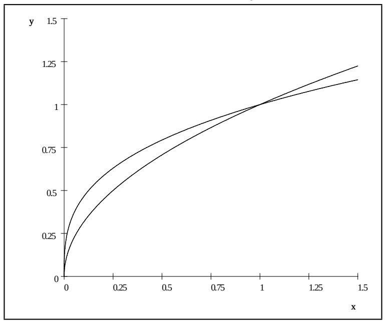
(B) $\frac{18}{210}$

(C) $\frac{1}{7}$

(D) $\frac{2}{5}$

(E) $\frac{13}{42}$

Sol: The curves $x = y^2$ and $x = y^3$ intersect at $(0,0)$ and $(1,1)$.



$$V = \int_0^1 \int_{y^3}^{y^2} \left(2x + y^2\right) dx dy = \int_0^1 \left(x^2 + xy^2\right)\Big|_{y^3}^{y^2} dy = \int_0^1 \left(y^4 + y^4 - y^6 - y^5\right)\Big| \, dy = \left(\frac{2y^5}{5} - \frac{y^7}{7} - \frac{y^6}{6}\right)\Bigg|_0^1$$
$$= \frac{2}{5} - \frac{1}{7} - \frac{1}{6} = \frac{19}{210}.$$

Q.20: (**Section 15.3–B**) The value of the iterated integral $\int_0^2 \int_{2x}^4 e^{y^2} dy dx$ is equal to:

(A) $\frac{1}{4}\left(e^{16} - 1\right)$

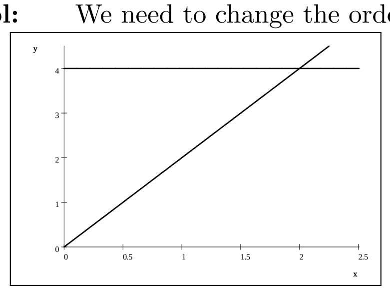
(B) $\frac{1}{2}\left(e^{16} + 1\right)$

(C) $\frac{1}{4}\left(e^{16} + 1\right)$

(D) $\frac{1}{2}\left(e^{16} - 1\right)$

(E) $\frac{1}{8}\left(e^{16} - 2\right)$

Sol: We need to change the order of integration. The region is $\{(x,y)\,|2x \le y \le 4,\ 0 \le x \le 2\}$



$$\int_0^2 \int_{2x}^4 e^{y^2} dy dx = \int_0^4 \int_0^{\frac{y}{2}} e^{y^2} dx dy = \int_0^4 \frac{y}{2} e^{y^2} dy = \left.\frac{e^{y^2}}{4}\right|_0^4 = \frac{e^{16} - 1}{4}.$$

Q.21: (**Section 15.4–B**) The value of the iterated integral $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dydx$ is equal to:

(A) $\frac{\pi}{4}\left(e^4-1\right)$

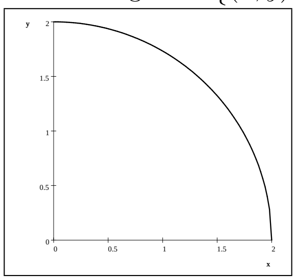
(B) $\frac{\pi}{2}\left(e^2-1\right)$

(C) $\frac{\pi}{4}\left(e^4+1\right)$

(D) $\frac{\pi e^2}{4}$

(E) $\frac{\pi e^{16}}{4}$

Sol: The region is $\left\{(x,y)\,|0 \le y \le \sqrt{4-x^2},\ 0 \le x \le 2\right\}$.



Changing into polar coordinates.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dydx = \int_0^{\frac{\pi}{2}} \int_0^2 e^{r^2} r\,dr\,d\theta = \frac{\pi}{2}\left(\frac{e^{r^2}}{2}\right)\Bigg|_0^2 = \frac{\pi}{4}\left(e^4-1\right).$$

Q.22: (**Section 15.7**) If volume of a tetrahedron formed by the plane $ax+y-z=4$ and the three coordinate planes is $\frac{16}{3}$, then value of a is:

(A) -2

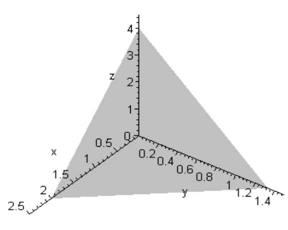
(B) 2

(C) -3

(D) 4

(E) 0

Sol:



$$V = \int_0^{\frac{4}{a}} \int_0^{4-ax} (ax+y-4)\,dydx = \int_0^{\frac{4}{a}} \left(axy + \frac{y^2}{2} - 4y\right)\Bigg|_0^{4-ax} dx = \int_0^{\frac{4}{a}} \left(ax\,(4-ax) + \frac{1}{2}(4-ax)^2 - 4\,(4-ax)\right) dx$$

$$= \int_0^{\frac{4}{a}} \left(4ax - a^2x^2 + 8 - 4ax + \frac{a^2x^2}{2} - 16 + 4ax\right) dx = \int_0^{\frac{4}{a}} \left(4ax - \frac{a^2x^2}{2} - 8\right) dx = \left(4a\frac{x^2}{2} - \frac{a^2x^3}{6} - 8x\right)\Big|_0^{\frac{4}{a}}$$
$$= \frac{32a}{a^2} - \frac{64a^2}{6a^3} - \frac{32}{a} = -\frac{32}{3a}. \text{ Thus } -\frac{32}{3a} = \frac{16}{3} \Rightarrow a = -2.$$

Q.23: (**Section 15.7**) The volume of the solid enclosed by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ is:

(A) $\frac{16}{3}$

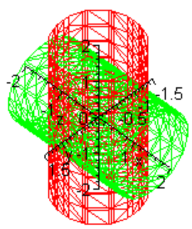
(B) 8

(C) $\frac{2}{3}$

(D) 4

(E) 1

Sol:



$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dzdydx = 2.2.2 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\sqrt{1-x^2}\right) dydx$$
$$= 8 \int_0^1 \left(1 - x^2\right) dx = 8\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{16}{3}.$$

Q.24: (**Section 15.8–B**) The value of $\iiint\limits_E \sqrt{x^2 + y^2 + z^2}dV$, where E is the solid that lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$ and above the $xy - plane$:

(A) $\frac{65\pi}{2}$

(B) $\frac{65\pi}{4}$

(C) $\frac{56\pi}{4}$

(D) $\frac{65\pi^2}{4}$

(E) $\frac{65\pi}{8}$

Sol: By changing into spherical coordinates,

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_2^3 \rho^3 \sin(\phi)\, d\rho d\phi d\theta = (2\pi)\left(\frac{\rho^4}{4}\right)\Big|_2^3 \left(-\cos(\phi)\right)\Big|_0^{\frac{\pi}{2}}$$
$$= (2\pi)\left(\frac{81-16}{4}\right)(1) = \frac{65}{2}\pi.$$

Q.25: (**Section 15.8–B**) The triple integral that gives the volume of the solid that lies inside the sphere $x^2+y^2+z^2=2$ and outside the cone $z^2=x^2+y^2$ is:

(A) $\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi d\rho d\phi d\theta$

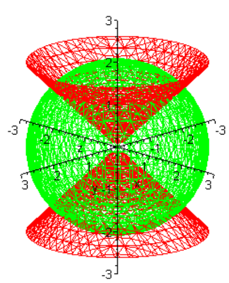
(B) $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi d\rho d\phi d\theta$

(C) $2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \rho^2 \sin\phi d\rho d\theta d\phi$

(D) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \rho^2 \sin\phi d\phi d\theta d\rho$

(E) $\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}}\int_{\frac{\pi}{2}}^{\frac{7\pi}{4}} \rho^2 \sin\phi d\phi d\rho d\theta$

Sol:



$$V = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi d\rho d\phi d\theta.$$