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Solution Final Exam for Math 201 (071), Sec 04, 06

Q.1: Find volume of the parallelepiped determined by the vectors $\mathbf{u} = \mathbf{i}+4\,\mathbf{j}-7\,\mathbf{k}$, $\mathbf{v} = 2\,\mathbf{i}-\,\mathbf{j}+4\,\mathbf{k}$, and $\mathbf{w} = \mathbf{i}-9\,\mathbf{j}+18\,\mathbf{k}$. Show all your work. **(10 pts)**

Sol: $V = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 1 & -9 & 18 \end{vmatrix} = 1\,(-18+38) - 4\,(38-4) - 7\,(-18+1) = 9$

Q.2: Find symmetric equation of the line of intersection of the planes $x+y-z=2$ and $3x-4y+5z=6$. Also find angle between these planes. **(5+5 pts)**

Sol: Let $z=0$, then $x+y=2$ and $3x-4y=6$ has solution $x=2$ and $y=0$.

The direction vector is $v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 3 & -4 & 5 \end{vmatrix} = i - 8j - 7k.$

Thus the symmetric equation of the line of intersection is $\dfrac{x-2}{1} = \dfrac{y}{-8} = \dfrac{z}{-7}$.

Angle between two planes is the same as angle between their normals.

Thus $\theta = \cos^{-1}\left(\dfrac{n_1 \cdot n_2}{|n_1|\,|n_2|}\right) = \cos^{-1}\left(\dfrac{3-4-5}{\sqrt{3}\sqrt{50}}\right).$

Q.3: Write the equation $\rho^2\left(\sin^2\phi - 3\cos^2\phi\right) = 1$ in cartesian coordinates. **(10 pts)**

Sol: $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta \ \Rightarrow x^2+y^2 = \rho^2\sin^2\phi$. Also $z = \rho\cos\phi$

Thus $\rho^2\sin^2\phi - 3\rho^2\cos^2\phi = 1 \ \Rightarrow x^2+y^2-3z^2 = 1.$

Q.4: Show that $f(x,y) = xe^{xy}$ is differentiable at the point $P(1,0)$ and find the linearization $L(x,y)$ of $f(x,y)$ at the point P. Use $L(x,y)$ to approximate $f(1.1,-0.1)$. **(10 pts)**

Sol: $f_x(x,y) = e^{xy} + xye^{xy}$ and $f_y(x,y) = x^2e^{xy}$. At $P(1,0)$, $f_x(1,0) = 1$ and $f_y(1,0) = 1$. Since both these derivatives are continuous, therefore f is differentiable at $P(1,0)$.

$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 1 + 1\,(x-1) + 1\,(y) = x+y.$

$f(1.1,-0.1) \approx L(1.1,-0.1) = 1.1 - 0.1 = 1.$

Q.5: The length l, width w and height h of a box are changing with time. If l is increasing at a rate of $3\ m/s$, w is decreasing at a rate of $2\ m/s$, and h is increasing at a rate of $1\ m/s$. Find the rate of change of the volume of the box when $l=10$, $w=8$, and $h=5$. **(10 pts)**

Sol: $V = lwh$ and

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\,\frac{dl}{dt} + lh\,\frac{dw}{dt} + lw\,\frac{dh}{dt} = (8)(5)(3) + (10)(5)(-2) + (10)(8)(1) = 100.$$

Q.6: Use chain rule to find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ if $z = w\sin^{-1}(uv)$, $u = r+s$, $v = s+t$, $w = t+r$. **(10 pts)**

Sol:
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial r}$$
$$= \sin^{-1}(uv)(1) + \frac{wv}{\sqrt{1-u^2v^2}}(1) + \frac{wu}{\sqrt{1-u^2v^2}}(0) = \sin^{-1}(uv) + \frac{wv}{\sqrt{1-u^2v^2}}$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s}$$
$$= \sin^{-1}(uv)(0) + \frac{wv}{\sqrt{1-u^2v^2}}(1) + \frac{wu}{\sqrt{1-u^2v^2}}(1) = \frac{w(v+u)}{\sqrt{1-u^2v^2}}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t}$$
$$= \sin^{-1}(uv)(1) + \frac{wv}{\sqrt{1-u^2v^2}}(0) + \frac{wu}{\sqrt{1-u^2v^2}}(1) = \sin^{-1}(uv) + \frac{wu}{\sqrt{1-u^2v^2}}.$$

Q.7: (a)Find the maximum rate of change of $f(x,y,z) = \tan(x+2y+3z)$ at the point $(-5,1,1)$. **(5 pts)**

Sol: (a) $\nabla f(x,y,z) = \left\langle \sec^2(x+2y+3z), 2\sec^2(x+2y+3z), 2\sec^2(x+2y+3z)\right\rangle$

and $\nabla f(-5,1,1) = \langle \sec^2(0), 2\sec^2(0), 2\sec^2(0)\rangle = \langle 1,2,3\rangle$.

The maximum rate of change occur in the direction $\nabla f(-5,1,1) = \langle 1,2,3\rangle$ and maximum rate of change is $|\nabla f(-5,1,1)| = \sqrt{1+4+9} = \sqrt{14}$.

(b) Write equations of the tangent plane and normal line to the surface $f(x,y,z) = 0$ at $(-5,1,1)$.**(5 pts)**

Equation of tangent plane at $(-5,1,1)$ is $1(x+5) + 2(y-1) + 3(z-1) = 0 \;\Rightarrow x + 2y + 3z = 0$.

Equation of normal line at $(-5,1,1)$ is $\dfrac{x+5}{1} = \dfrac{y-1}{2} = \dfrac{z-1}{3}$.

Q.8: Find, if any, the local maximum, local maximum and saddle point(s) of the function $f(x,y) = 2x^4 + 2y^4 - 8xy + 12$. **(10 pts)**

Sol: $f_x = 8x^3 - 8y$ and $f_y = 8y^3 - 8x$. $f_x = 0 \;\Rightarrow x^3 = y$ and $f_y = 0 \;\Rightarrow y^3 = x$.

$x^9 = x \;\Rightarrow x\left(x^8 - 1\right) = 0 \;\Rightarrow x = 0$ or $x = \pm 1$.

If $x = 0$, then $y = 0$. If $x = \pm 1$ then $y = \pm 1$. Thus the critical points are $(0,0), (1,1), (-1,-1)$.

Now $f_{xx} = 24x^2,\; f_{yy} = 24y^2,\; f_{xy} = -8$.

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (24)^2 x^2y^2 - 64$.

$D(0,0) = -64 < 0$. So the point $(0,0)$ is a saddle point.

$D(1,1) = (24)^2 - 64 = 512 > 0$ and $f_{xx}(1,1) = 24 > 0 \;\Rightarrow (1,1)$ is a local minimum.

$D(-1,-1) = (24)^2 - 64 = 512 > 0$ and $f_{xx}(-1,-1) = 24 > 0 \;\Rightarrow (-1,-1)$ is a local minimum.

Q.9: Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y,z) = 8x + 6y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 26$. **(10 pts)**

Sol: Let $g(x,y,z) = x^2 + y^2 + z^2$.

Then $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \;\Rightarrow 8 = 2\lambda x,\; 6 = 2\lambda y,\; 2 = 2\lambda z \;\Rightarrow x = \dfrac{4}{\lambda},\; y = \dfrac{3}{\lambda},\; z = \dfrac{1}{\lambda}$.

$g(x,y,z) = 26 \;\Rightarrow \dfrac{16}{\lambda^2} + \dfrac{9}{\lambda^2} + \dfrac{1}{\lambda^2} = 26 \;\Rightarrow \lambda = \pm 1$.

For $\lambda = 1,\; x = 4, y = 3, z = 1$ and for $\lambda = -1,\; x = -4, y = -3, z = -1$.

$f(4,3,1) = 32 + 18 + 2 = 52$ and $f(-4,-3,-1) = -32 + -18 - 2 = -52$.

Thus maximum value of f, subject to the given constraint, is 54 and minimum value is -54.

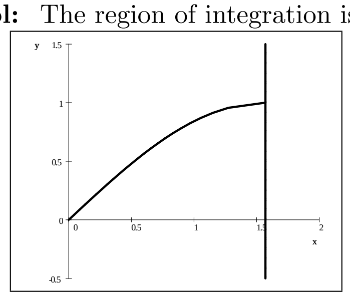
Q.10: Find volume of the solid enclosed by the surface $z = e^y \sin x + e^x \cos y$ and the planes $x = 0,\; x = \pi,\; y = 0,\; y = \dfrac{\pi}{2}$, and $z = 0$. **(10 pts)**

Sol: $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\pi} (e^y \sin x + e^x \cos y)\,dxdy = \int_0^{\frac{\pi}{2}} \left(-e^y \cos x + e^x \cos y\right)\Big|_0^{\pi} dy$

$\displaystyle= \int_0^{\frac{\pi}{2}} (2e^y + e^{\pi}\cos y - \cos y)\,dy = \left(2e^y + e^{\pi}\sin y - \sin y\right)\Big|_0^{\frac{\pi}{2}} = 2e^{\frac{\pi}{2}} + e^{\pi} - 3.$

Q.11: Evaluate the integral $\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\; dxdy$ (Hint: Change order of integration). **(10 pts)**

Sol: The region of integration is



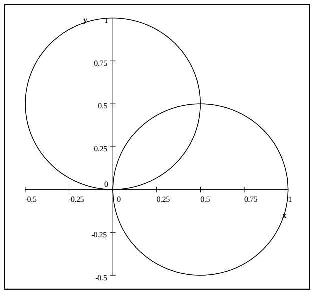
$\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\; dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dydx = \int_0^{\frac{\pi}{2}} \sin x \cos x\sqrt{1+\cos^2 x}\,dx$

Use substitution $u = 1 + \cos^2 x$, then $du = -2\sin x \cos x dx$ and $u = 2$ when $x = 0,\; u = 1$ when $x = \dfrac{\pi}{2}$.

Thus we get $\displaystyle -\frac{1}{2}\int_2^1 \sqrt{u}du = \frac{1}{2}\int_1^2 (u)^{\frac{1}{2}}\,du = \frac{1}{2}\frac{2}{3}\,(u)^{\frac{3}{2}}\Big|_1^2 = \frac{1}{3}\left(2^{\frac{3}{2}} - 1\right).$

Q.12: Use double integrals in polar coordinates to find the area bounded by the circles $r = \sin\theta$ and $r = \cos\theta$. **(10 pts)**

Sol: The two circles intersect at $\theta = \dfrac{\pi}{4}$



Since region bounded by two curves from 0 to $\dfrac{\pi}{4}$ is the same as from $\dfrac{\pi}{4}$ to $\dfrac{\pi}{2}$. Therefore, because of symmetry, we can use

$$2\int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{2}d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi}{8}-\frac{1}{4}.$$

Q.13: Find volume of the region bounded by the coordinate planes and the plane $2x+y+z=2$. **(10 pts)**

Sol: The coordinate planes are $z=0,\ y=0,\ x=0$.

$$\int_0^1\int_0^{2-2x}\int_0^{2-2x-y} dz\,dy\,dx = \int_0^1\int_0^{2-2x}(2-2x-y)\,dy\,dx = \int_0^1 \left(2y-2xy-\frac{1}{2}y^2\right)\Big|_0^{2-2x} dx$$

$$= \int_0^1 \left(2(2-2x)-2x(2-2x)-\frac{1}{2}(2-2x)^2\right)dx$$

$$= \int_0^1 \left(4-4x-4x+4x^2-2-2x^2+4x\right)dx$$

$$= \int_0^1 \left(2x^2-4x+2\right)dx = \left(2\frac{x^3}{3}-2x^2+2x\right)\Big|_0^1 = \frac{2}{3}.$$

Q.14: Evaluate $\iiint\limits_E x^2 dV$, where E is the solid region bounded by the hemispheres $z=\sqrt{4-x^2-y^2}$ and $z=\sqrt{9-x^2-y^2}$ and the $xy-plane$.

Sol: $z=\sqrt{4-x^2-y^2} \Rightarrow z^2 = 4-x^2-y^2$ a sphere of radius 2 and $z=\sqrt{9-x^2-y^2}$ is a sphere of radius 3.

Since solid region is bounded by $xy-plane$, therefore $0\le\theta\le\pi$ and $0\le\phi\le\dfrac{\pi}{2}$.

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_2^3 \rho^2\sin^2\phi\cos^2\theta\ \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\frac{\pi}{2}}\sin^3\phi\,d\phi\ \int_0^{\pi}\cos^2\theta\,d\theta\ \int_2^3\rho^4\,d\rho$$

$$= \int_0^{\frac{\pi}{2}}\sin\phi\left(1-\cos^2\phi\right)d\phi\ \int_0^{\pi}\frac{1+\cos 2\theta}{2}d\theta\ \left(\frac{\rho^5}{5}\right)\Big|_2^3$$

$$= \int_0^{\frac{\pi}{2}}\sin\phi\,d\phi - \int_0^{\frac{\pi}{2}}\sin\phi\cos^2\phi\,d\phi\ \left(\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi}\right)\ \left(\frac{3^5}{5}-\frac{2^5}{5}\right)$$

$$= \left[(-\cos\phi)\big|_0^{\frac{\pi}{2}} + \left(\frac{\cos^3\phi}{3}\right)\Big|_0^{\frac{\pi}{2}}\right]\ \left(\frac{\pi}{2}\right)\left(\frac{243}{5}-\frac{32}{5}\right)$$

$$= \left[1-\frac{1}{3}\right]\left(\frac{\pi}{2}\right)\left(\frac{211}{5}\right) = \pi\frac{211}{15}$$

Q.15: Find volume of the region bounded by the surfaces $z=x^2+y^2$ and $z=2-x^2-y^2$

Sol: The intersection of these two surfaces is $x^2+y^2=2-x^2-y^2 \Rightarrow x^2+y^2=1$ circle of radius 1. Also $z=x^2+y^2=r^2$ and $z=2-x^2-y^2=2-r^2$.

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\left(2-r^2-r^2\right)dr\,d\theta = \int_0^{2\pi}d\theta\ \int_0^1\left(2r-2r^3\right)dr = 2\pi\ \left(r^2-\frac{2r^4}{4}\right)\Big|_0^1 = \pi.$$