King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Solution: Exam # 3, Math 302 – 03

Note: Show all your work to get full credit.

Q.1: Evaluate $\iint_\Sigma (\nabla\times\mathbf{F})\cdot\hat{n}\,d\sigma$ or $\oint_C F\cdot dR$ for $F = yx^2\mathbf{i} - xy^2\mathbf{j} + z^2\mathbf{k}$, and Σ is the hemisphere $x^2+y^2+z^2 = 9$, $z \geq 0$.

Sol: $F = [yx^2, -xy^2, z^2]$, $\varphi = x^2+y^2+z^2-9$

$$\nabla\times F = \begin{bmatrix} 0 \\ 0 \\ -x^2-y^2 \end{bmatrix}, \qquad \nabla\varphi = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \ \|\nabla\varphi\|_2 = \sqrt{4x^2+4y^2+4z^2} = \sqrt{4\,(9)} = 6$$

$$\hat{n} = \tfrac{1}{3}[x,y,z]$$

$$d\sigma = \sqrt{1+\left(-\frac{x}{z}\right)^2+\left(-\frac{y}{z}\right)^2}dA = \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}dA = \sqrt{\frac{1}{z^2}\left(x^2+y^2+z^2\right)} = \frac{3}{z}dA$$

$$\nabla\times F\cdot\hat{n} = \frac{-\left(x^2+y^2\right)z}{3}, \text{ and } \nabla\times F\cdot\hat{n}\ d\sigma = -\left(x^2+y^2\right)dA$$

$$\iint_\Sigma(\nabla\times\mathbf{F})\cdot\hat{n}\,d\sigma = \iint_D -\left(x^2+y^2\right)dA = \int_0^{2\pi}\int_0^3 -r^2 r\,dr\,d\theta = -\frac{81}{2}\pi.$$

OR:

Consider the courve as $x^2+y^2=9$, and $z=0$. The parametric equations are

$x = 3\cos(t)$, $y = 3\sin(t)$, $z=0$, $0\leq t\leq 2\pi$,

$$F = \begin{bmatrix} 27\cos^2 t\sin t & -27\cos t\sin^2 t & 0 \end{bmatrix},\ R = [x,y,z] = \begin{bmatrix} 3\cos t & 3\sin t & 0 \end{bmatrix},.$$

$$\frac{dR}{dt} = \begin{bmatrix} -3\sin t & 3\cos t & 0 \end{bmatrix},$$

$$F\cdot\frac{dR}{dt} = \left(-81\cos^2 t\sin^2 t - 81\cos^2 t\sin^2 t\right) = -\frac{81}{2}\left(4\cos^2 t\sin^2 t\right) = -\frac{81}{2}\left(\sin(2t)\right)^2$$
$$-\frac{81}{2}\sin^2(2t) = -\frac{81}{4}2\sin^2(2t) = -\frac{81}{4}\left(1-\cos(4t)\right)$$

$$\oint_C F\cdot dR = \int_0^{2\pi} -\frac{81}{4}\left(1-\cos(4t)\right)dt = -\frac{81}{2}\pi.$$

Q.2: Let $z,\ w$, and u be complex numbers. Prove that these numbers form vertices of an equilateral triangle if and only if $z^2+w^2+u^2 = zw+wu+uz$.

Sol: As vectors, $w-z = (u-w)\,e^{\frac{2\pi}{3}i}$

and $u-w = (z-u)\,e^{\frac{2\pi}{3}i}$

Divide and cross multiply

$$\frac{w-z}{u-w} = \frac{u-w}{z-u}$$

$\Rightarrow z^2+w^2+u^2 = zw+wu+uz$

Q.3: Find the limit $\lim_{n\to\infty}\left\{\dfrac{2n^2+3n^2 i}{(2n+1)(n-2)}\right\}$.

Sol: $\lim\limits_{n\to\infty} \dfrac{2n^2+3n^2 i}{(2n+1)(n-2)} = \lim\limits_{n\to\infty} \dfrac{2n^2}{2n^2-3n-2} + i \lim\limits_{n\to\infty} \dfrac{3n^2}{2n^2-3n-2} = 1+\dfrac{3}{2}i$

Q.4: Determine all points where Cauchy-Riemann equations are satisfied for $f(z) = \dfrac{-4z^2+1}{z}$ and also determine the points where $f(z)$ is differentiable.

Sol: $z = x+iy,\ f(z) = \dfrac{-4z^2+1}{z} = -4z + \dfrac{1}{z} = \dfrac{1}{x+iy} - 4iy - 4x = \dfrac{x-iy}{x^2+y^2} - 4iy - 4x$

$$= -4x + \frac{x}{x^2+y^2} + i\left(-4y - \frac{y}{x^2+y^2}\right)$$

Define $u(x,y) = -4x + \dfrac{x}{x^2+y^2},\ v(x,y) = -4y - \dfrac{y}{x^2+y^2}$

$$\frac{\partial u(x,y)}{\partial x} = \frac{1}{x^2+y^2} - 2\frac{x^2}{x^4+y^4+2x^2y^2} - 4,$$

$$\frac{\partial u(x,y)}{\partial y} = -2x\frac{y}{x^4+y^4+2x^2y^2},$$

$$\frac{\partial v(x,y)}{\partial x} = 2x\frac{y}{x^4+y^4+2x^2y^2},$$

$$\frac{\partial v(x,y)}{\partial y} = 2\frac{y^2}{x^4+y^4+2x^2y^2} - \frac{1}{x^2+y^2} - 4$$

Thus $\dfrac{\partial u(x,y)}{\partial x} = \dfrac{\partial v(x,y)}{\partial y}$ is true, and $\dfrac{\partial v(x,y)}{\partial x} = -\dfrac{\partial u(x,y)}{\partial y}$ is true.

Q.5: Determine the center and radius of convergence of the series $\sum\limits_{n=0}^{\infty}\left(\dfrac{n^n}{(n+1)^n}\right)(z+2-5i)^n$.

Sol: Let $R = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^{n+1}}{(n+2)^{n+1}}(z+2-5i)^{n+1}\dfrac{(n+1)^n}{n^n}\dfrac{1}{(z+2-5i)^n}\right|$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+2)^{n+1}}\frac{(n+1)^n}{n^n}(z+2-5i)\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{n+1}\left(\frac{n+1}{n}\right)^n|(z+2-5i)|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n+1+1}\right)^{n+1}\left(\frac{n+1}{n}\right)^n|(z+2-5i)|$$

$$= \lim_{n\to\infty}\frac{1}{\left(\frac{n+1+1}{n+1}\right)^{n+1}}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n|(z+2-5i)|$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\left(1+\frac{1}{n}\right)^n|(z+2-5i)|$$

$$= \frac{1}{e}e\,|(z+2-5i))| = |(z+2-5i))| < 1 \qquad \text{Using} \quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

Thus the center of convergence is $-2+5i$ and radius of convergence is 1.

Q.6: Find $u(x,y)$ and $v(x,y)$ for $f(z) = e^{\frac{1}{z}}$ such that $f(z) = u + iv$.

Sol: $z = x+iy$, and $\dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$

$$f(z) = e^{\frac{1}{z}} = e^{\frac{x}{x^2+y^2}}.e^{-i\frac{y}{x^2+y^2}} = e^{\frac{x}{x^2+y^2}}\left(\cos\left(\frac{y}{x^2+y^2} - i\sin\left(\frac{y}{x^2+y^2}\right)\right)\right)$$

$$u(x,y) = e^{\frac{x}{x^2+y^2}}\left(\cos\frac{y}{x^2+y^2}\right)$$

$$v(x,y) = -ie^{\frac{x}{x^2+y^2}}\sin\left(\frac{y}{x^2+y^2}\right)$$

Q.7: Determine all the values of $(2+3i)^{\,1-i}$.

Sol: $(2+3i)^{1-i}$=$\mathrm{e}^{(1-i)\ln(2+3i)}$

$$\ln(2+3i) = \ln\left(\sqrt{13}\right) + i\left(\tan^{-1}\left(\frac{3}{2}\right) + n\pi\right), \text{ where } n \text{ is any integer.}$$

$$(1-i)\left(\ln\left(\sqrt{13}\right) + i\left(\tan^{-1}\left(\frac{3}{2}\right) + n\pi\right)\right) = \ln\left(\sqrt{13}\right) + \left(\tan^{-1}\left(\frac{3}{2}\right) + n\pi\right) + i\left(\left(\tan^{-1}\left(\frac{3}{2}\right) + n\pi\right) - \ln\left(\sqrt{13}\right)\right)$$

For simplicity let $X = \ln\left(\sqrt{13}\right) + \left(\tan^{-1}\left(\frac{3}{2}\right) + n\pi\right)$

$$Y = \left(\tan^{-1}\left(\frac{3}{2}\right) + n\pi\right) - \ln\left(\sqrt{13}\right)$$

Thus $e^{(1-i)\ln(2+3i)} = e^X\left(\cos(Y) + i\sin(Y)\right)$

Q.8: Determine the three cube roots of 8.

Sol: Let $z = 8 + 0i = 8e^{i(0+2n\pi)}$, where n is any integer.

$z^{\frac{1}{3}} = 2e^{i\left(\frac{2n\pi}{3}\right)} = 2\left(\cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)\right)$, $n = 0, 1, 2$ for distinct values.

$z_1 = 2,\ z_2 = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = i\sqrt{3} - 1$

$z_2 = 2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -i\sqrt{3} - 1$