King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences

Solution: Exam # 2, Math 302 – 03

Name:... .....Serial #:..........................

Note: Show all your work to get full credit.

Q.1: Let $x(t) = \sin(t),\ y(t) = \cos(t),\ z(t) = t,\ 0 \leq t \leq 2\pi$. Write the position vector and the tangent vector for the curve. Also find the length function.

Sol: $F(t) = [\sin(t), \cos(t), t]$

$T(t) = \dfrac{dF(t)}{dt} = [\cos t, -\sin t, 1]$.

$\|T(t)\| = \sqrt{\left(|\cos t|^2 + |\sin t|^2 + 1\right)} = \sqrt{2}$.

Q.2: Find the directional derivative of the function $\varphi(x, y, z) = 1 - x^2 - y^2 - xyz$ in the direction of the vector $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Also find equation of the tangent plane to the surface $\varphi(x, y, z) = 0$, at the point $(1, 1, -1)$.

Sol: $\varphi(x, y, z) = 1 - x^2 - y^2 - xyz$

$u = [1, -1, 1]$

$\|u\| = \sqrt{3}$ and unit vector is $\hat{u} = \frac{u}{\|u\|} = \frac{1}{\sqrt{3}}(i - j + k)$

$\nabla\varphi(x, y, z) = (-2x - yz, -2y - xz, -xy)$

$D_u\varphi = \nabla\varphi(x, y, z) \cdot \hat{u} = \frac{1}{\sqrt{3}}(-2x - yz + 2y + xz - xy)$

$\nabla\varphi(x, y, z)|_{(1,1,-1)} = (-1, -1, -1)$

$-(x - 1) - (y - 1) - (z + 1) = 0$

$x + y + z = 1$

Q.3: Let $\mathbf{F} = \sinh(x^2 - z)\mathbf{i} + 2xy\mathbf{j} + (z^2 - y^2)\mathbf{k}$. Find Divergence of $\mathbf{F}$ and Curl of $\mathbf{F}$.

Sol: $F = [\sinh(x^2 - z), 2xy, z^2 - y^2]$

$\nabla \cdot F = 2\left(\cosh\left(x^2 - z\right)\right)x + 2x + 2z$

$\nabla \times F = \left[-2y, -\cosh\left(x^2 - z\right), 2y\right]$

Q.4: Sow that $\nabla \times (\nabla\varphi) = \mathbf{0}$, and $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Q.5: Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = \sin(x)\mathbf{i} + 2z\mathbf{j} - \mathbf{k}$ and C is given by $x = t,\ y = t^2,\ z = t^3$, for $0 \leq t \leq 3$.

Sol: $F(x, y, z) = [\sin(x), 2z, -1]$

$x = t$

$y = t^2$

$z = t^3$

$F(x, y, z) = [\sin t, 2t^3, -1]$

$R(t) = [t, t^2, t^3]$

$F(x, y, z) \cdot \frac{d}{dt}R(t) = \sin t + 4t^4 - 3t^2$

$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^3 (\sin t + 4t^4 - 3t^2)\, dt = -\cos 3 + \frac{842}{5} =$

Q.6: Use Green's Theorem to evaluate the integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = x^2y\mathbf{i} - xy^2\mathbf{j}$ and C is the boundary of the region $x^2 + y^2 \leq 4$, $y \geq 0$.

Sol: $F(x, y) = [x^2y, -xy^2]$

$\frac{\partial}{\partial x}(-xy^2) = -y^2$

$\frac{\partial}{\partial y}(x^2y) = x^2$

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_D (-y^2 - x^2)\, dA = -\int_0^\pi \int_0^2 r^2 r\,dr\,d\theta = -4\pi$$

Q.7: Evaluate the surface integral $\iint_\Sigma f(x, y, z)\, d\sigma$, where $f(x, y, z) = z$ and Σ is the cone $x = \sqrt{y^2 + z^2}$, for $y^2 + z^2 \leq 9$.

Sol: $f(x, y, z) = z$, $x = \sqrt{y^2 + z^2}$, $y^2 + z^2 \leq 9$

$$\frac{\partial x}{\partial y} = \frac{y}{\sqrt{y^2 + z^2}}, \quad \frac{\partial x}{\partial z} = \frac{z}{\sqrt{y^2 + z^2}},$$

$$d\sigma = \sqrt{1 + \left(\frac{y}{\sqrt{y^2 + z^2}}\right)^2 + \left(\frac{z}{\sqrt{y^2 + z^2}}\right)^2}\, dA = \sqrt{\frac{y^2}{y^2 + z^2} + \frac{z^2}{y^2 + z^2} + 1}\, dA = \sqrt{2}\, dA$$

$$\iint_\Sigma f(x, y, z)\, d\sigma = \iint_D z\sqrt{2}\, dA = \int_0^{2\pi} \int_0^3 r\cos(\theta)\, r\,dr\,d\theta = 0$$

Q.8: Evaluate the surface integral $\iint_\Sigma f(x, y, z)\, d\sigma$, where $f(x, y, z) = x$ and Σ is part of the plane $5x + 2y + z = 10$ in the first octant.

Sol: $f(x, y, z) = x$, $z = 10 - 5x - 2y$,

$$\frac{\partial z}{\partial x} = -5, \quad \frac{\partial z}{\partial y} = -2$$

$$d\sigma = \sqrt{1 + 25 + 4}\, dA = \sqrt{30}\, dA$$

$$\iint_\Sigma f(x, y, z)\, d\sigma = \iint_D x\sqrt{30}\, dA = \int_0^5 \int_0^{\frac{10-2y}{5}} x\,dx\,dy = \frac{10}{3}, \quad \text{OR} \int_0^2 \int_0^{\frac{10-5x}{2}} x\,dy\,dx = \frac{10}{3}$$

Q.9: Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = (-4\cosh(xy) - 4xy\sinh(xy))\,\mathbf{i} - 4x^2\sinh(xy)\mathbf{j}$, and C is any path from $(1, 1)$ to $(2, 2)$. (Hint: find potential function φ for $\mathbf{F}$ if it is conservative)

Sol: $F = [-4\cosh(xy) - 4xy\sinh(xy), -4x^2\sinh(xy)]$,

$$\frac{\partial(-4x^2\sinh(xy))}{\partial x} = -8x\sinh xy - 4x^2y\cosh xy,$$

$$\frac{\partial(-4\cosh(xy) - 4xy\sinh(xy))}{\partial y} = -8x\sinh xy - 4x^2y\cosh xy,$$

Thus F is conservative and there exist a potential function φ such that $F = \nabla\varphi$

$$\frac{\partial\varphi}{\partial x} = -4\cosh(xy) - 4xy\sinh(xy), \text{ and } \frac{\partial\varphi}{\partial y} = -4x^2\sinh(xy)$$

Integrate $\frac{\partial\varphi}{\partial y} = -4x^2\sinh(xy)$ wrt y $\Rightarrow \varphi = -4x\cosh(xy) + K(x)$

Diff $\varphi = -4x\cosh(xy) + K(x)$ wrt x $\Rightarrow \frac{\partial\varphi}{\partial x} = -4\cosh xy - 4xy\sinh xy + \frac{\partial K(x)}{\partial x}$

Comparing with $\frac{\partial\varphi}{\partial x} = -4\cosh(xy) - 4xy\sinh(xy) \Rightarrow \frac{\partial K(x)}{\partial x} = 0$, and $K(x) = C = 0$.

Hence $\varphi(x,y) = -4x\cosh(xy)$ and $\oint\limits_C \mathbf{F}\cdot d\mathbf{R} = \varphi(2,2) - \varphi(1,1) = 4\cosh 1 - 8\cosh 4$

Q.10: Calculate one side of the Gauss's Divergence Theorm for the vector field $\mathbf{F} = (x-y)\,\mathbf{i} + (y-4xz)\,\mathbf{j} + xz\,\mathbf{k}$, Σ is the rectangular box bounded by the coordinate planes $x=0,\ y=0,\ z=0,$ and by the planes $x=3,\ y=4,\ z=5.$

Sol: $F = [x-y, y-4xz, xz],$

$\nabla\cdot F = x+2$

$$\iiint\limits_M \nabla\cdot F dV = \int_0^5\int_0^4\int_0^3 (x+2)\,dxdydz = 210$$